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GEODETIC PROBLEMS AND SATELLITE ORBITS

by W. H. Guier

Lectures I, II, III, IV and V

GEODETIC PROBLEMS AND SATELLITE ORBITS

W. H. Guier

LECTURE I

INTRODUCTORY REMARKS

The title for this series of five lectures is "Geodetic Problems and Satellite Orbits". Clearly, when tracking satellites, our only real knowledge that certain problems exist in the area of geodesy is through a study of the satellite tracking data, noting that present knowledge of geodesy is inadequate to theoretically describe and/or predict the detailed time dependence of the received tracking data. For this reason, the principal topic to be discussed in this series is the effect of geodetic errors on the time dependence of satellite tracking data as received by a tracking station located on the surface of the Earth from a near-earth satellite. These geodetic errors fall into two categories, geodetic errors which effect the location of the tracking station on the surface of the Earth and geodetic errors which effect the motion of the satellite (and therefore its position at some given value of the time). Consequently, subsidiary topics which shall be discussed are:

1. Methods for specifying the motion of a tracking station in inertial space, given the usual geodetic measurements available for a point on the earth's surface,
2. The motion of a near-earth satellite when influenced by the various harmonics of the earth's gravity field (geopotential), and

3. The functional dependence of various types of tracking data upon the trajectories of the station and satellite in inertial space.

These topics do not cover many problem areas relating to satellite motion and the accurate reception and tabulation of tracking data. Such problem areas, while important from the standpoint of achieving accurate prediction of the trajectories of satellites, can reasonable well be divorced from the geodetic problem areas. Consequently, this series of lectures will assume a rather narrow definition of the word geodetic problems - namely problems associated with the science of determining the shape and size of the Earth and its gravity field.

Fundamentally, the procedure of determining the orbit of a satellite can be considered as the process of assuming the satellite to be under the influence of a known force field and then using the tracking data to determine which solution to the equations of motion one should choose. By this I mean the following. Assuming for the moment that the forces acting on the satellite are known, an infinity of solutions to the differential equations of motion exist until boundary conditions are imposed - such as values for the initial position and velocity of the satellite at some chosen epoch. The tracking data is used to determine as accurately as possible these initial conditions. Consequently, errors in satellite orbits can arise from errors in the forces that act on the satellite and errors in the computed boundary conditions. Within the area of interest of these lectures, the geopotential is considered as

the sole source of error in the satellite forces, and tracking station location errors the sole source in obtaining errored boundary conditions.

In principle, errors in the location of tracking stations can be discussed entirely separate from errors in the satellite forces. However, in practice, complete separation of the two sources of errors cannot be made. The primary reason is that the accurate determining of the station location depends in practice upon a knowledge of the geopotential (near the earth's surface) and consequently errors in the geopotential introduce errors in both the station and satellite trajectories in inertial space. Another important reason is because, to zeroth order, satellite tracking data provides information on the position and/or velocity of the satellite relative to that of the station. Consequently, it is frequently difficult to accurately separate orbit errors into those directly related to the station position and those directly related to the satellite motion.

It can be seen from the above discussion that central to the determination of station positions and satellite orbits is an accurate specification of the earth's gravitational force field, and I shall now briefly discuss a representation for the gravity field of the Earth. We chose the sign convention such that the force is given by $+\text{grad } U$, where U is the gravitational potential of the Earth. It is common to express this potential as an expansion in surface harmonics so that:

$$U(R, \varphi, \lambda) = \frac{K}{R} \left\{ 1 + \sum_{n=2}^{\infty} \left(\frac{R_0}{R} \right)^n [J_n P_n(\sin \varphi) + \sum_{m=1}^{\infty} P_n^m(\sin \varphi) (C_n^m \cos m \lambda + S_n^m \sin m \lambda)] \right\}$$

where

$$\begin{aligned} K &= \text{gravity force constant (km}^3/\text{sec}^2), \\ R_0 &= \text{mean equatorial radius of Earth (km),} \\ R &= \text{geocentric radius (km),} \\ \varphi &= \text{geocentric latitude (rad),} \\ \lambda &= \text{geocentric longitude (rad),} \end{aligned}$$

and where

$$P_n^m(Z) = (1 - Z^2)^{\frac{m}{2}} \frac{d^m}{dZ^m} P_n(Z).$$

The geocentric coordinates R , φ , and λ have their origin located at the center of gravity of the Earth. The geocentric latitude is measured from a plane which passes through the earth's C.G. and is normal to the earth's spin axis. The geocentric longitude is measured positive eastward from the plane containing the spin axis and a special marker at the Observatory in Greenwich, England - the so called Greenwich meridian. Since the origin of this coordinate system is at the center of gravity of the Earth, it follows that $J_1 = C_1^1 = S_1^1 = 0$. To the accuracy that we will consider in these lectures we may assume that there is sufficient energy dissipation that the earth's spin axis is the principal axis of the largest moment of inertia of the Earth and therefore we may assume that the spin axis passes through the earth's C.G. Consequently, in the above expansion for the geopotential we also may take $C_2^1 = S_2^1 = 0$. Finally, to the accuracy which we shall consider, we may assume that the earth's gravitational field is time independent and that the spin axis, equatorial

plane, and Greenwich meridian are fixed with respect to the crust or surface of the Earth. Except for some relatively minor considerations when discussing the geoid, we shall not be interested in the gravitational field below the physical surface of the Earth.

Corresponding to the geocentric coordinates R , ϕ , and λ there is a natural right-handed cartesian coordinate system fixed with respect to the Earth. This is shown in Figure 1. The Greenwich meridian is the X-Z plane and the equatorial plane coincides with the X-Y plane.

Because of the earth's rotation it is not convenient to describe the satellite motion in a coordinate system which is fixed with respect to the earth's crust. A very natural coordinate system for the satellite motion is one which has its Z-axis coinciding with the earth's spin axis and its X and Y axis approximately fixed relative to inertial space (fixed relative to the celestial sphere). This inertial coordinate system and its relationship with the earth fixed cartesian system is shown in Figure 2. Very briefly, the inertial system is defined in the following way.¹ The apparent motion of the sun around the Earth approximately lies in a plane called the ecliptic plane. The intersection of this plane with the earth's equatorial plane defines a line which is approximately fixed in inertial space. We take the positive X-axis of the inertial system as the direction of this line of intersection going from the C.G. of the Earth in that direction where the sun crosses the equatorial plane going from south to north. This direction is known to the astronomer as the First Line of Aries. This coordinate system is called the True Equatorial System of Date to denote that it is defined by the direction of the instantaneous spin axis of the Earth and the intersection of the

instantaneous equatorial and ecliptic planes. This system experiences small accelerations due to the fact that the earth's spin axis precesses and nutates relative to inertial space and the apparent motion of the sun around the Earth does not lie exactly in a fixed plane. However, for our purposes this coordinate system is a sufficient approximation to an inertial system and for coordinate systems which are more accurately inertial you may refer to reference 1.

It is inevitable that other coordinate systems must be introduced when discussing the location of a tracking station on the surface of the Earth. This is because all surveying is done on the surface of the Earth and it is most natural to define coordinate systems which are surface coordinate systems. I shall now briefly discuss the various geodetic coordinates required to locate a tracking station referring you to references 2 and 3 for details.

A surface from which a natural surface coordinate system can be developed is one of the equipotential surfaces for the Earth. If this equipotential surface is chosen to coincide with mean sea level (average height of the sea surface when corrected for tides, weather effects, etc.) the surface is known as the geoid. This surface, by definition, is everywhere normal to the direction of the force of gravity, and all measurements of relative height are most naturally referenced to the geoid. When over land the geoid is not measurable in as straightforward a manner as one might think. Clearly many areas will have the geoid located below the physical surface of the Earth. When this is the case it is necessary to correct for the gravitating mass that is above the geoid when using gravity measurements to determine the geoid. Correcting for this mass

inevitably involves assumptions as to the density, inhomogeneities, etc., for the crustal mass, and for clarity one refers to the co-geoid^{2,3} rather than the geoid when discussing the determination of an equipotential surface over land masses. To the accuracy required for these lectures however we may assume that the geoid and co-geoid are coincident and, consistent with the previous assumptions, we may assume that the geoid is time independent.

The shape of the geoid is sufficiently complex that it is inconvenient to use in computations. For this reason it is common to use an oblate spheroid (ellipse of revolution) which approximately follows the geoid in specifying the geodetic coordinates of a station. Figure 3 shows a meridional section of a spheroid with the pertinent quantities used to define the spheroid and the coordinates of a point on the surface of the spheroid. A spheroid, being an ellipse of revolution, has its surface defined when its semi-major axis and eccentricity are defined. In practice the flattening, f , is given instead of the eccentricity and is related to the eccentricity by the formula: $\frac{1}{f} = 1 + \epsilon^2$. The latitude and longitude of a station are always referred to the spheroid. The geodetic latitude, ϕ_G , is defined by dropping a perpendicular to the surface of the spheroid and noting the angle of intersection of this normal with the equatorial plane. Consequently, the cartesian coordinates ζ_G, Z_O in the meridian containing the station are (see Figure 3).

$$\zeta_G = \frac{a}{\sqrt{1 + (1 - 1/f)^2 \tan^2 \phi_G}} = \sqrt{x_O^2 + y_O^2},$$

$$Z_O = (1 - 1/f)^2 \zeta_G \tan \phi_G.$$

The longitude is, of course, related to the cartesian coordinates X_o , Y_o by $\lambda_G = \tan^{-1} Y_o/X_o$.

In specifying the orientation of a spheroid with respect to the spin axis and center of gravity of the Earth the intent is normally to have the semi-minor axis coincide with the spin axis and the semi-major axis lying in the equatorial plane with the center of the spheroid at the center of gravity of the Earth. In practice the specification of this orientation is done at the surface of the Earth at a point which is denoted as the datum point. This implies that the spheroid is oriented to the geoid at a point on the surface of the Earth which does not coincide with either the spheroid or the geoid. Such a connection is subject to measurement errors such that any given spheroid associated with a major surveyed area does not in fact coincide with the center of gravity of the Earth and the earth's spin axis.

With the advent of satellites and their use for improving the force field of the Earth it is becoming common practice to define a world wide survey system or datum which has its spheroid, by definition, orientated correctly with respect to the center of gravity of the Earth and its spin axis. For example, the current NASA World Datum has as its semi-major axis and flattening

$$R_o = 6378.166 \text{ kilometers}$$

$$f_o = 298.24$$

With such a definition for the orientation of the spheroid it then becomes a straightforward procedure to state the coordinates of the geoid and the various geodetic coordinates of the tracking station relative to this spheroid and to give transformation formulas for obtaining the geocentric coordinates of a station. Of course when using such a world wide datum it is necessary to obtain transformation formulas from the datum of a major surveyed network such as the North American Datum to the World Datum. Such transformations normally assume that the spheroid for the local datum has its axes parallel to the axes of the world datum spheroid so that a translation only is needed to transform from one spheroid to the other.

Before proceeding further, I shall now briefly show that to first order in the flattening, f , a spheroid approximates an equipotential surface for the Earth. This proof depends upon the experimental fact that

$$J_2 = O(1/f)$$

$$J_n, C_n^m, S_n^m = O(1/f^2), \quad n > 2.$$

The proof proceeds in the following manner. For any point on the spheroid

$$X_o, Y_o, Z_o, R_o = \sqrt{X_o^2 + Y_o^2 + Z_o^2},$$

let

$$\sin \varphi = \frac{Z_o}{R_o}, \quad \cos \varphi = \frac{\sqrt{X_o^2 + Y_o^2}}{R_o},$$

a = semi-major axis of spheroid,

f = flattening.

Then:

$$\frac{R_o^2}{a^2} \left[\cos^2 \varphi + \frac{\sin^2 \varphi}{(1 - 1/f)^2} \right] = 1$$

For any point rigidly connected to the Earth, the measured gravitational potential will be the sum of the gravitational potential, U , as measured in inertial space and a potential whose gradient yields the centrifugal force arising from the earth's rotation. Letting this earth-fixed potential be ψ and noting that all coefficients in the expansion for U are $O(1/f^2)$ except J_2 :

$$\psi = \frac{K}{R} \left[1 + \frac{J_2}{2} \left(3 \frac{Z^2}{R^2} - 1 \right) + \frac{\omega_E^2 R (X^2 + Y^2)}{2K} + O(1/f^2) \right],$$

where ω_E = angular rotation rate of Earth (rad/sec). We consider now the potential, ψ_o , for any point X_o, Y_o, Z_o on the spheroid. From the above equations:

$$\psi_o = \frac{K}{a} \left\{ 1 - \frac{J_2}{2} + \frac{\omega_E^2 a^3}{2K} + \sin^2 \varphi \left[1/f + \frac{3}{2} J_2 - \frac{\omega_E^2 a^3}{2K} \right] + O(1/f^2) \right\},$$

where it has been noted that:

$$\frac{\omega_E^2 a^3}{2K} = O(1/f).$$

Thus, letting

$$1/f = -\frac{3}{2} J_2 + \frac{\omega_E^2 a^3}{2K} + O(J_2^2),$$

we have

$$\psi_0 = \frac{K}{a} \left\{ 1 - \frac{J_2}{2} + \frac{\omega_E^2 a^3}{2K} + O(1/f^2) \right\},$$

which is a constant to $O(1/f)$.

The above proof indicates that the geoid (more properly the co-geoid) will not differ markedly from a properly defined spheroid. Consequently, the spheroid provides a convenient base for specifying quantitatively the geoid. This is done by specifying the geoidal height, $H(\varphi_G, \lambda_G)$ for any given geodetic latitude, φ_G , and longitude, λ_G , as defined on the spheroid. This relationship is shown in Figure 4A where it can be seen that any point X_G, Y_G, Z_G on the geoid is related to the geodetic latitude and longitude by the formulas:

$$\begin{aligned} X_G &= (\zeta_G + H \cos \varphi_G) \cos \lambda_G, \\ Y_G &= (\zeta_G + H \cos \varphi_G) \sin \lambda_G, \\ Z_G &= (1 - 1/f)^2 \zeta_G \tan \varphi_G + H \sin \varphi_G. \end{aligned}$$

We are now ready to include the remaining geodetic quantities needed to specify the geocentric location of a tracking station. Those quantities which have not yet been discussed are (in order of importance):

h = elevation of station above geoid (measured normal to geoid),
 ξ = deflection of local vertical in meridian (positive north),
 ζ = deflection of local vertical in prime meridian (positive east),
 $\delta X, \delta Y, \delta Z$ = position of center of spheroid associated with local survey
 relative to center of world-wide (NASA) spheroid.

Figure 4B shows schematically the first of these three quantities in relation to the geoid and spheroid. The last three are self explanatory.

Without further discussion I shall now give the final computational procedure for determining a station's geocentric cartesian coordinates given the geodetic quantities that I have just previously discussed. For further details I refer you to references 2 and 3.

$$\zeta_L = \frac{a}{\sqrt{1 + (1 - 1/f) \tan^2 \varphi_G}}, \quad a, f = \text{semi-major axis and flattening for local spheroid.}$$

$$X_R = [\zeta_L + (H + h) \cos \varphi_G] \cos \lambda_G - h[\xi \sin \varphi_G \cos \lambda_G + \eta \cos \varphi_G \sin \lambda_G]$$

+ δX + second order in ξ and η ,

$$Y_R = [\zeta_L + (H + h) \cos \varphi_G] \sin \lambda_G - h[\xi \sin \varphi_G \sin \lambda_G - \eta \cos \varphi_G \cos \lambda_G]$$

+ δY + second order in ξ and η ,

$$Z_R = [(1 - 1/f)^2 \zeta_L + (H + h) \cos \varphi_G] \tan \varphi_G + h \xi \cos \varphi_G$$

+ δZ + second order in ξ and η .

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1. Newton, R. R., "Astronomy for the Non-Astronomer", IRE Transactions on Space Electronics and Telemetry, Vol. Set-6, No. 1, March 1960.
2. Bomford, Brigadier G., "Geodesy", Clarendon Press, (1952).
3. Heiskanen, W. A. and Vening Meinesz, F. A., "The Earth and its Gravity Field", McGraw-Hill Publishing Company, (1958).

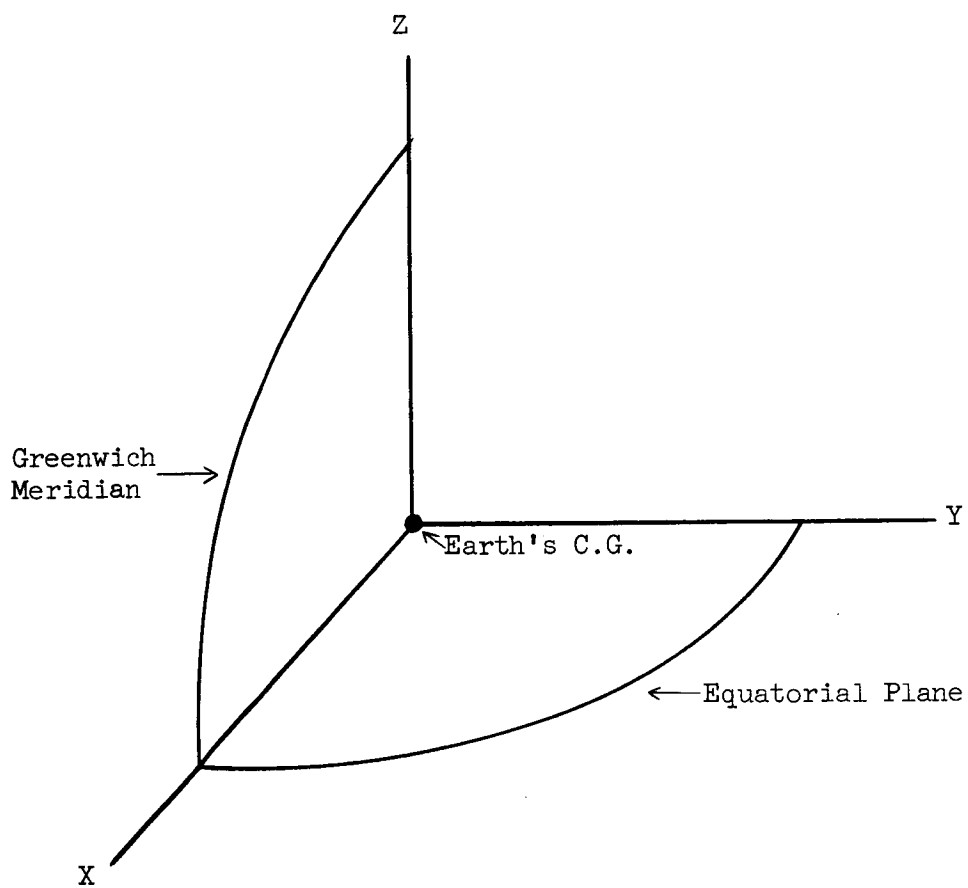


Figure 1.

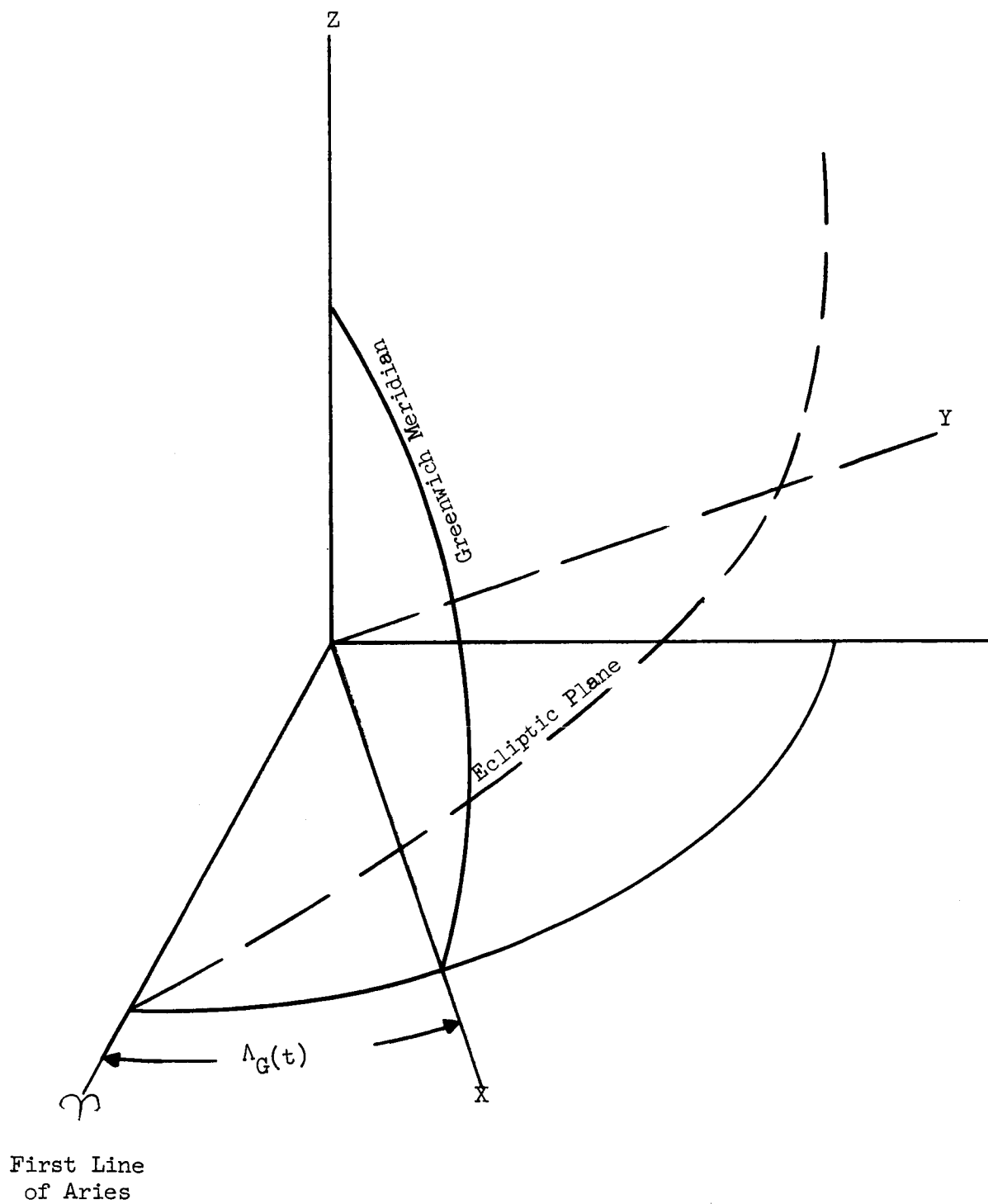


Figure 2

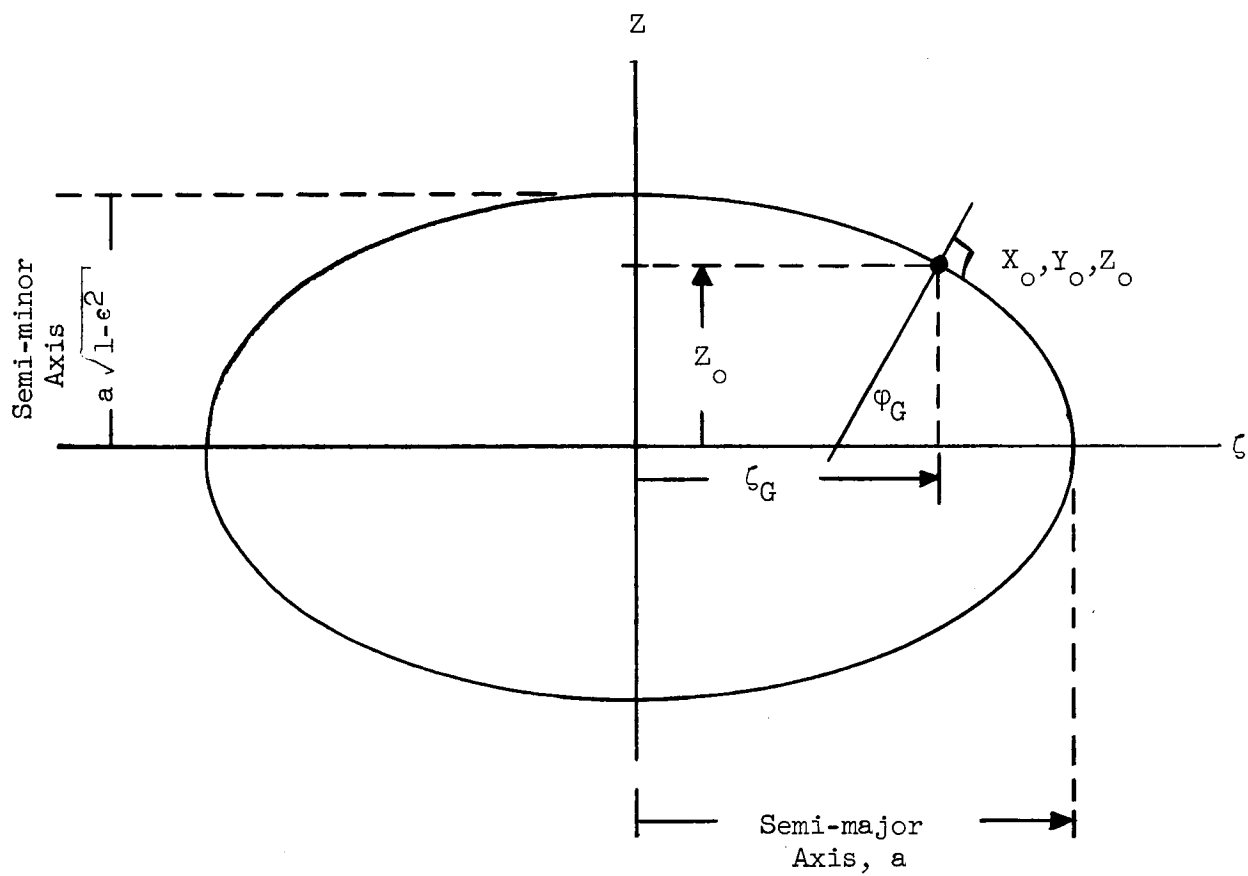


Figure 3

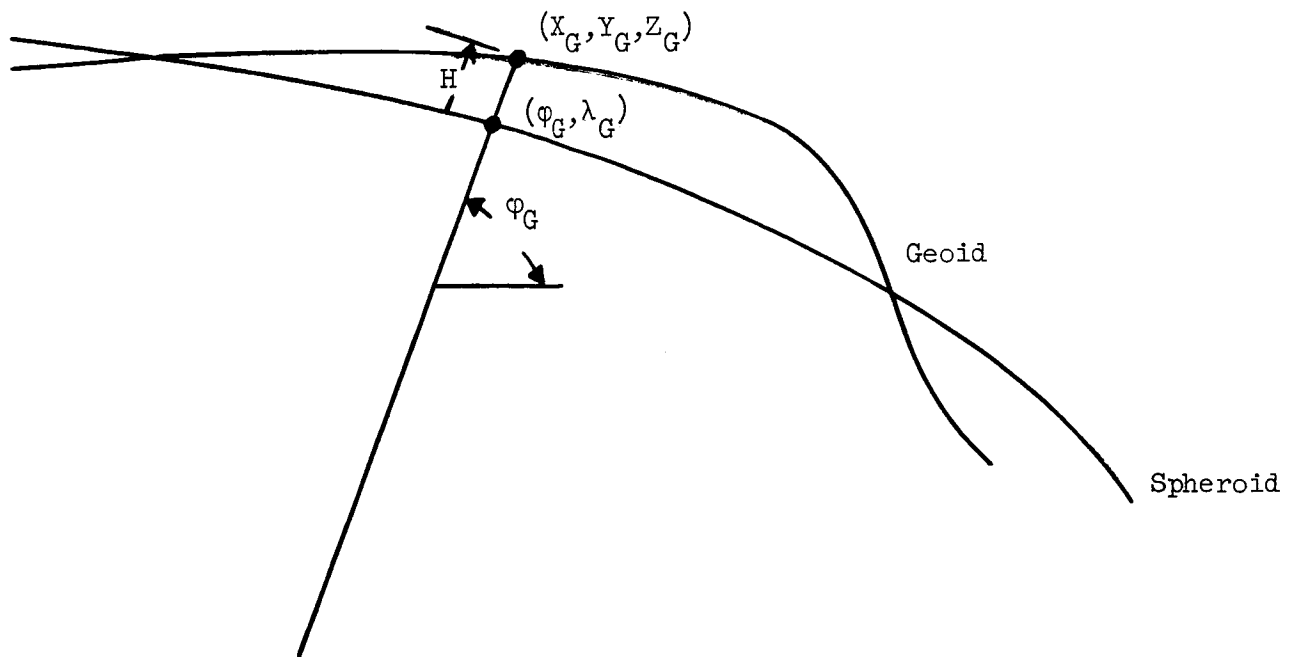


Figure 4A

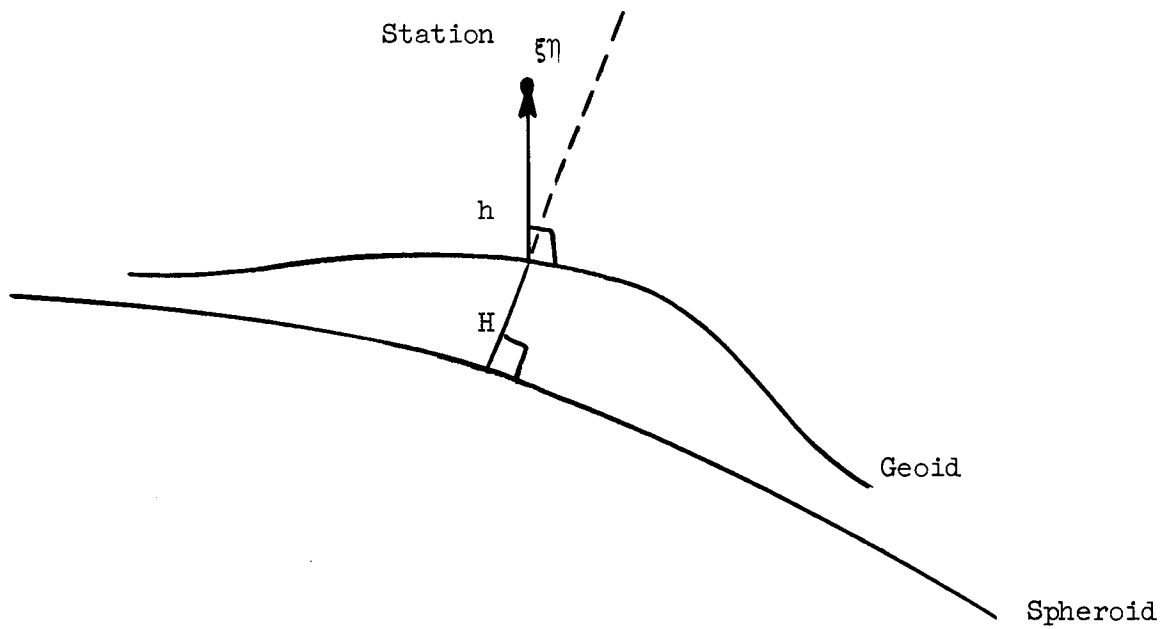


Figure 4B

LECTURE II

INTRODUCTORY REMARKS CONTINUED

In Lecture I we briefly considered a suitable representation for the geopotential and its relation to methods for locating a tracking station on the surface of the Earth. I now wish to turn our attention to the motion of a satellite under the influence of the geopotential and to present some working formulas relating the geometry of the satellite relative to such a station, which will be needed in the future lectures when we consider in more detail the effect of errors in the location of the tracking station and in the satellite motion.

Generally when we speak of a satellite orbit we imply the ability to compute (to some acceptable accuracy) the position of the satellite as a function of time in inertial space (for example the True Equatorial System of Date). The computation of such a satellite ephemeris clearly implies that a well defined force field has been assumed to be acting on the satellite, and satellite tracking data has been used to determine the orbit parameters (initial boundary conditions) for the solution of the differential equations of motion for the satellite.

Since we are primarily interested in the geodetic aspects of satellites and their motion I shall make the following restrictive assumptions to simplify the analysis which will be presented in the following lectures.

A. Assumptions Concerning Satellite Orbits.

1. Satellite motion

- a. non-relativistic approximation to equations of motion,
- b. near-earth satellites with small eccentricity (satellite altitude not less than about 1000 km and eccentricity $e \lesssim .05$).

2. Satellite forces not considered^{*}

- a. non-gravitational in origin,
 - 1) air drag
 - 2) radiation pressure
 - 3) electromagnetic
- b. non-static and extra-terrestrial gravitational forces,
 - 1) Sun, Moon, other planets, etc.
 - 2) earth's body and sea tides.

In addition to these assumptions we presume that we have at our disposal a world-wide net work of tracking stations together with the necessary data links and computer programs to establish (or track) the satellite to an accuracy limited by the accuracy of the geopotential and station locations assumed and the accuracy of the experimental tracking data. To further simplify our considerations I shall assume that there are negligible errors in the experimental tracking data. In particular I assume:

^{*}See for example reference 4 for a discussion of their effects.

B. Assumptions concerning experimental tracking data.

1. Signal propagation errors due to atmosphere are not considered,
 - a. ionospheric and tropospheric refraction (scintillation if optical data),
 - b. ducting, skip propagation, etc.
2. Experimental instrumentation errors are negligible,
 - a. misalignment and poorly calibrated tracking instruments,
 - b. "front-end" receiver (detector) noise
 - c. errors in transmission and formatting of data.

There are four fundamental measurements that are commonly made during the time that a satellite is above the horizon of a tracking station. These are:

1. Vector slant range

$$\vec{\rho}(t) \equiv \vec{r}_s(t) - \vec{r}_R(t)$$
2. Scalar slant range

$$\rho(t) = |\vec{\rho}(t)|$$
3. Slant range unit vector

$$\hat{\rho}(t) = \vec{\rho}(t)/\rho(t)$$
4. Scalar slant range rate

$$\dot{\rho}(t) = \frac{d}{dt} \rho(t) = \hat{\rho}(t) \cdot \dot{\vec{\rho}}(t)$$

where:

$\vec{r}_s(t), \dot{\vec{r}}_s(t)$ = satellite position and velocity in True Equatorial System of Date

$\vec{r}_R(t), \dot{\vec{r}}_R(t)$ = tracking station position and velocity in True Equatorial System of Date.

The slant range vector is typically the type of data taken by a tracking radar using the narrow beam pattern of the antenna to measure the slant range unit vector and its range (time of flight) instrumentation to measure the scalar slant range. Some radar tracking systems measure only the scalar slant range recognizing that the operating frequency is too low to accurately define angles. Optical tracking, of course, measures the slant range unit vector that is, right ascension and declination or azimuth and elevation. Finally tracking systems exist which use the measurement of the radio Doppler shift to make direct measurement of the scalar slant range rate. Some installations measure the slant range vector as well as the scalar slant range.

Clearly, the above types of data involve various combinations of quantities directly related to the relative geometry between the satellite and station during the time that the satellite is above the station's horizon. The remainder of this lecture will be devoted to presenting notation, convenient coordinate systems, and expressions relating the various quantities associated with the relative geometry between the satellite and station.

Let

t_c = time of closest approach of satellite to station,

t_R = time of satellite rise above station's horizon,

t_s = time of satellite set below station's horizon,

$\beta(t)$ = satellite argument of latitude,

$\Delta\beta_o = \beta(t_s) - \beta(t_c) \approx \beta(t_c) - \beta(t_R),$

E_ℓ, A_z = elevation and azimuth of satellite at t_c .

Figures 5, 6, and 7 show the geometry of the pass and presents a convenient coordinate system in which to consider the motion of the satellite relative to the station. This coordinate system is fixed in the satellite inertial space and has its coordinate axes defined at the time of closest approach, t_c . The Z-axis is defined to be the direction of the instantaneous angular momentum vector of the satellite at t_c . In Figure 5, the X-axis is defined as that line of intersection between the equatorial plane and the plane normal to the Z-axis and which contains the satellite position at t_c . The Y-axis is chosen such that the X, Y, Z coordinate system is a right-handed system. Clearly, the X-Y plane is the osculating plane of the orbit at the time of closest approach.

Figure 6 presents in more detail the pass geometry at the time of closest approach where the H-axis passes through the position of the satellite at t_c . Figure 7 presents the geometry of the pass projected on the X-Y plane and where the new coordinate axis, L, has been introduced to make the H, L, Z coordinate system a right-handed system. In Figure 7, the satellite position relative to its position at the time of closest approach is approximately shown with the change in the argument of latitude being denoted by $\Delta\beta$. (For simplicity the motion of the station during the time of the pass has been approximated as zero for clarity. The coordinate system which will be of primary interest to us in the following lectures is the H, L, Z coordinate system presented in these three figures.

The usual definitions for the elevation, E_ℓ , and azimuth, A_z , are inconvenient when deriving general formulas valid for all possible

paths of satellites past a given tracking station. For example, if a satellite passes through the zenith of the station the azimuth makes a discontinuous change of 180° . Two quantities directly related to the azimuth and elevation are much more conveniently used in such derivations. These have been denoted as the "pseudo azimuth", a_z , and "pseudo elevation", e . Figures 8A, 8B, and 8C show the relationships between the normally defined azimuth and elevation and the pseudo azimuth and elevation. It can be seen that the pseudo azimuth and elevation are obtained by altering the quadrants in which the azimuth and elevation lie such that there is continuity in changing from one type of pass geometry to another. For example, referring to Figure 6, the pseudo elevation is indicated and (for the case shown) can be seen to be identical with the normally defined elevation. This pseudo elevation will remain continuous as the vector ρ_z decreases through zero and goes negative, at which time the pseudo elevation increases beyond 90° . From Figures 8A and 8B it can also be seen that as ρ_z goes negative there is no discontinuity in the value for the pseudo azimuth.

In the lectures to follow the effects of the errors will be considered to first order. Consequently, the coefficients multiplying these errors need be derived only to a crude accuracy. For example, to sufficient accuracy the change in the station position during the time of the pass can be neglected in the expression for the slant range when it is involved in expressions which have been expanded to first order in the errors. Those relations which will be needed in the following lectures are now briefly summarized to the required accuracy. For details, I refer you to reference 5.

Let

$$r_R = |\vec{r}_R(t_c)|, \quad r_s = |\vec{r}_s(t_c)|,$$

$$r_{R,s} = r_R/r_s, \quad \rho_s = \rho(t_c)/r_s.$$

Then, from Figure 7,

$$r_{R,s}^2 = 1 + \rho_s^2 - 2\rho_s \cos \theta,$$

and

$$\sin \theta = r_{R,s} \sin (\pi/2 + e) = r_{R,s} \cos e.$$

These two formulas may be rearranged to yield:

$$\begin{aligned} \rho_s &= \cos \theta - \sqrt{r_{R,s}^2 - \sin^2 \theta}, \\ &= \frac{1 - r_{R,s}^2}{\sqrt{1 - r_{R,s}^2 \cos^2 e} + r_{R,s} \sin e} \end{aligned}$$

Neglecting the station motion in inertial space, to zeroth order the slant range vector in the H, L, Z coordinate system becomes

$$\vec{\rho}(t) = \begin{pmatrix} \rho_H(t) \\ \rho_L(t) \\ \rho_Z(t) \end{pmatrix},$$

$$= r_s \begin{pmatrix} \rho_s \cos \theta - 1 + \cos \Delta\beta(t) \\ \sin \Delta\beta(t) \\ - \rho_s \sin \theta \end{pmatrix} + \text{first order},$$

where,

$$\Delta\beta(t) = \dot{\beta}(t_c) (t - t_c) + O(\epsilon).$$

Finally, defining the quantities

$$\alpha_s = 1 - \rho_s \cos \theta,$$

$$C(t) = 1 - \cos \Delta\beta(t),$$

$$\vec{\rho}(t) = r_s \begin{pmatrix} 1 - \alpha_s - C(t) \\ \sin \Delta\beta(t) \\ - \rho_s \sin \theta \end{pmatrix} + \text{first order},$$

with

$$\rho(t) = \sqrt{\vec{\rho}(t) \cdot \vec{\rho}(t)} = r_s \sqrt{\rho_s^2 + 2 \alpha_s C(t)} + \text{first order}.$$

Suggested References for Lecture II.

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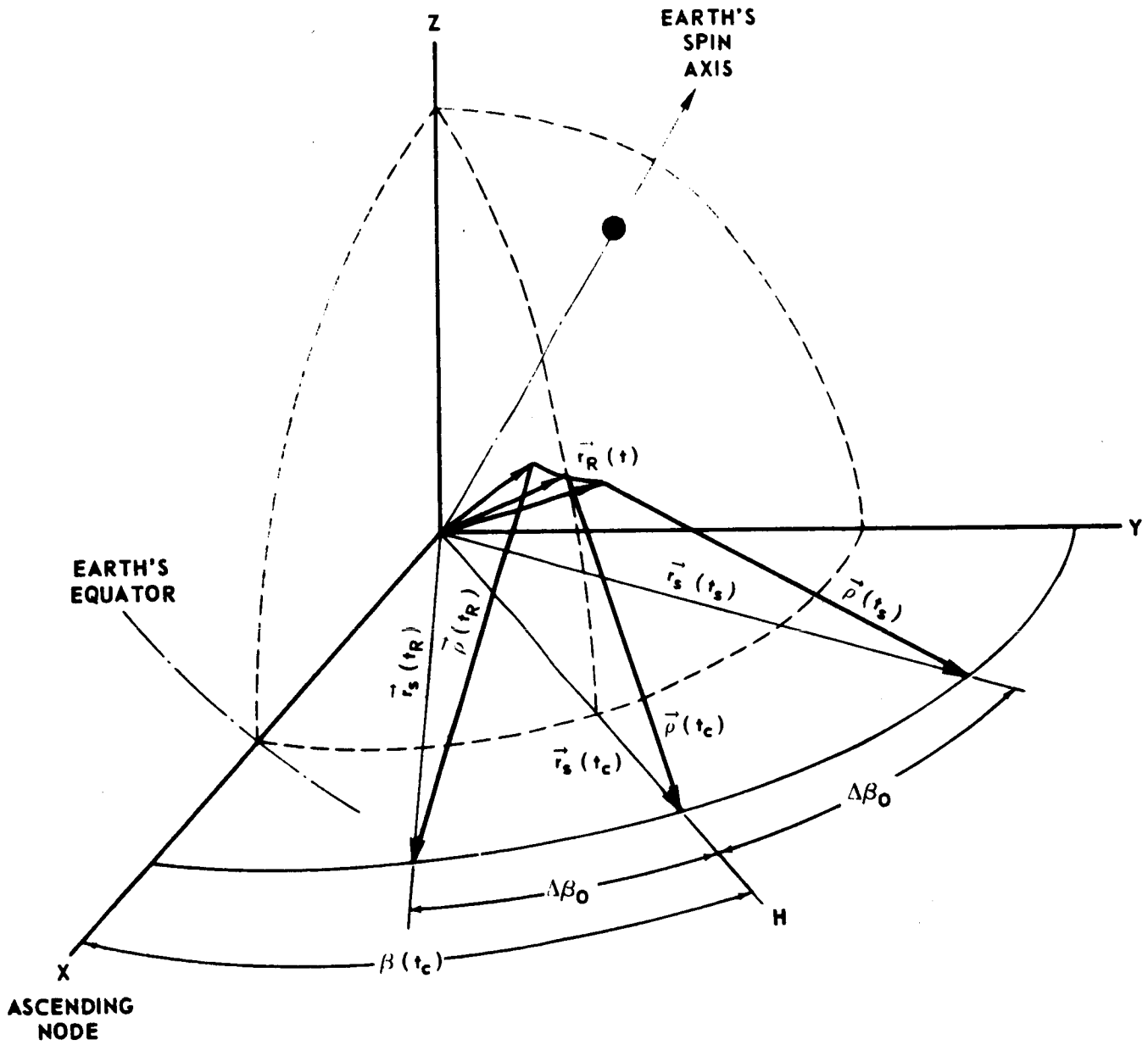


Fig. 5 Geometry During Satellite Pass ($x-y$ plane = Orbital Plane)

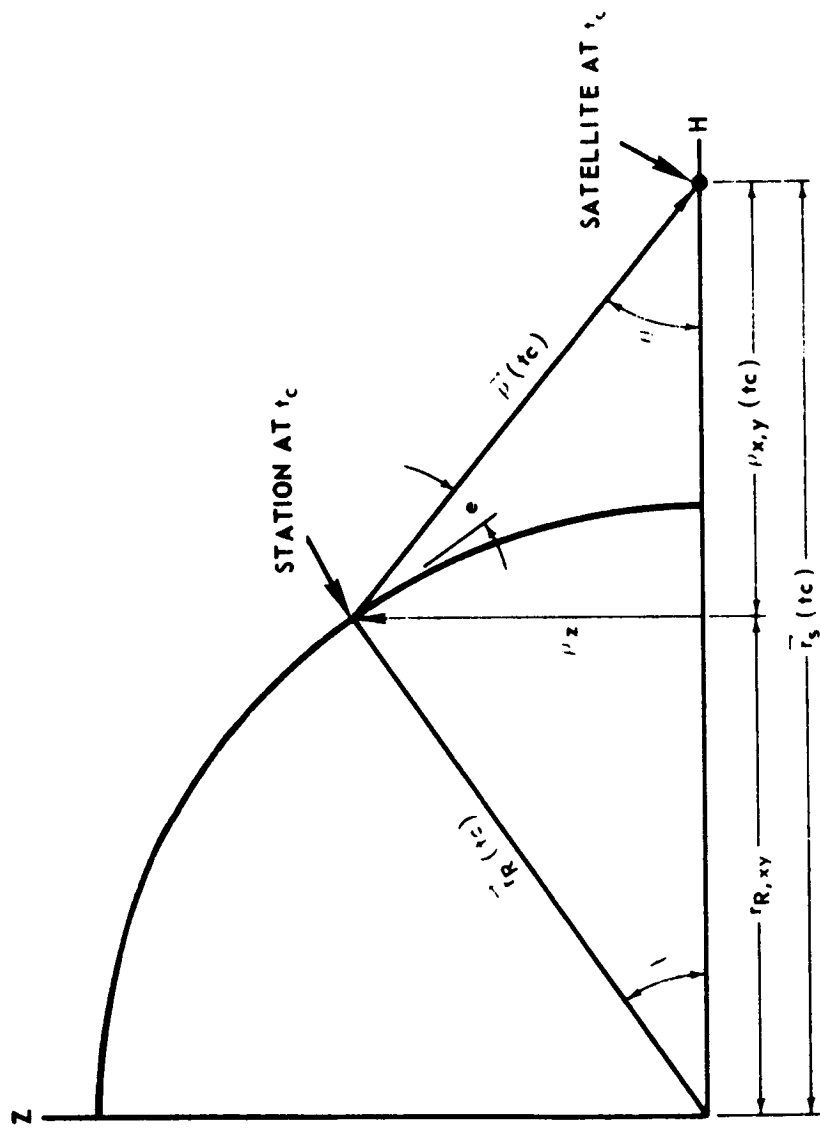


Fig. 6 Geometry at Time of Minimum Slant Range (H-Z Plane, Satellite motion into page)

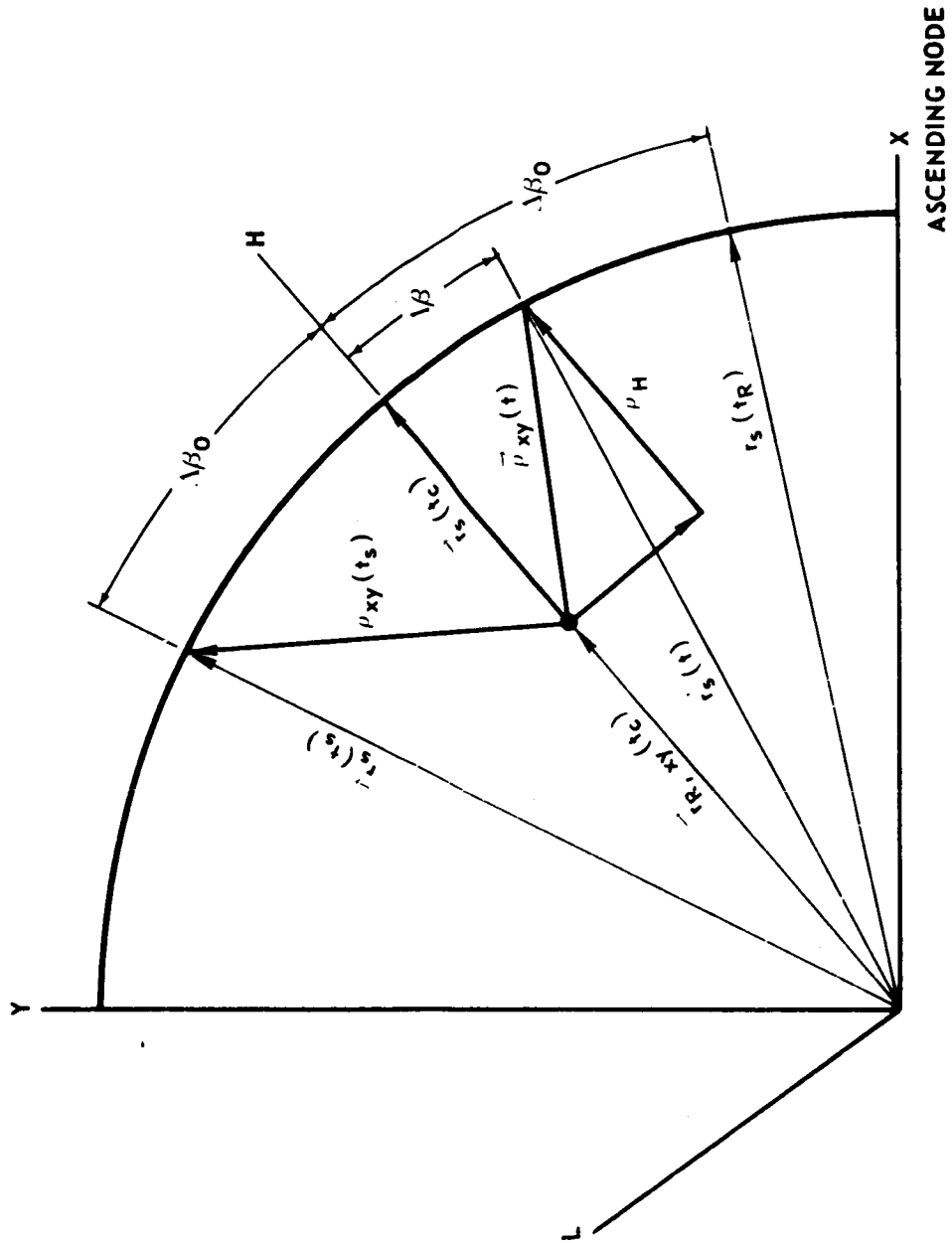


Fig. 7 Geometry of Pass (Orbital Plane)

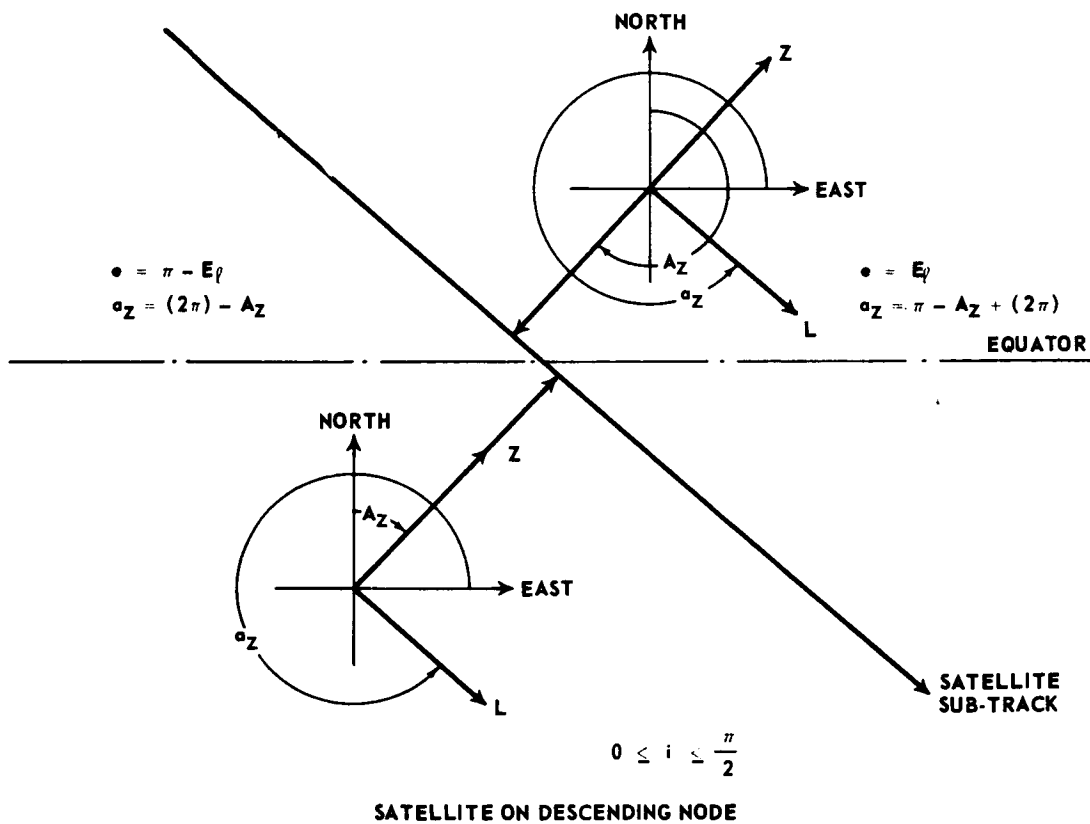
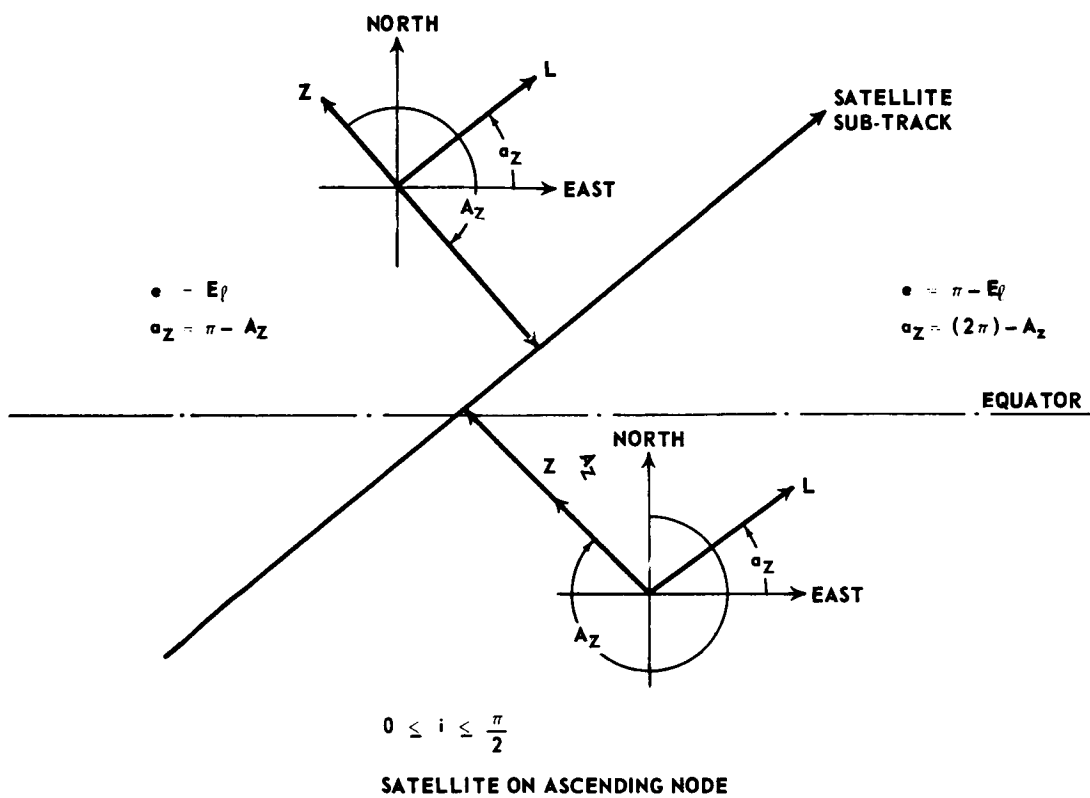


Fig. 8A Pseudo Elevation and Azimuth (Advance Satellite Motion)

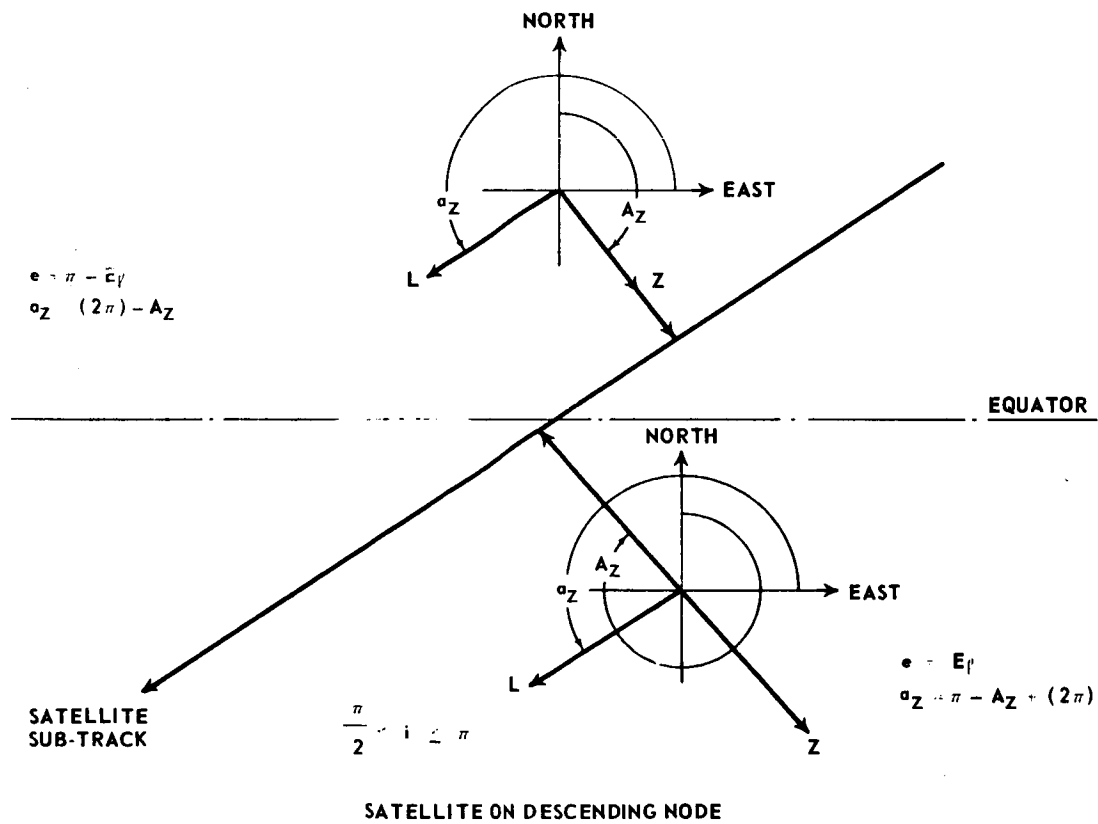
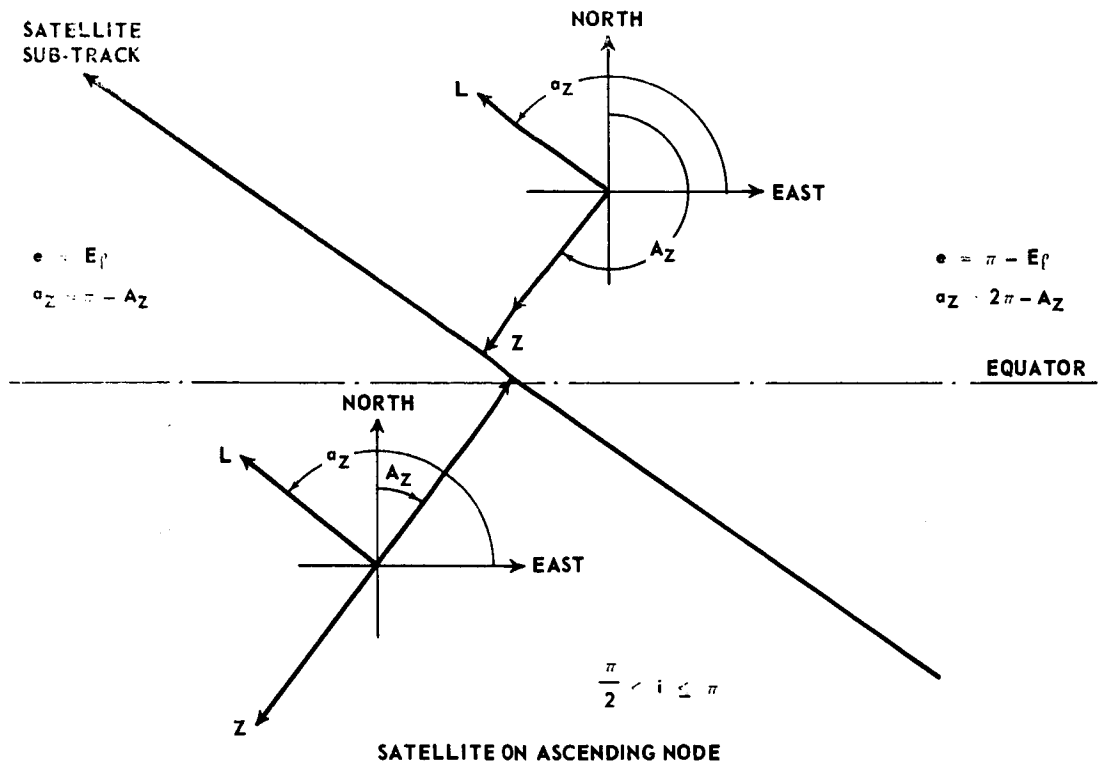


Fig. 8B Pseudo Elevation and Azimuth (Retrograde Satellite Motion)

Figure 8c

PSEUDO ELEVATION AND AZIMUTH

Azimuth	Satellite Inclination	
	$0 \leq i \leq \pi/2$	$\pi/2 < i \leq \pi$
$0 \leq A_z < \pi/2$	$e = \pi - E_\ell$	$e = E_\ell$
$\pi/2 \leq A_z < \pi$	$a_z = -A_z$	$a_z = \pi - A_z$
$\pi \leq A_z < \frac{3\pi}{2}$	$e = E_\ell$	$e = \pi - E_\ell$
$\frac{3\pi}{2} \leq A_z < 2\pi$	$a_z = \pi - A_z$	$a_z = -A_z$

E_ℓ = Elevation
 A_z = Azimuth

e = Pseudo Elevation
 a_z = Pseudo Azimuth

LECTURE III

STATION AND SATELLITE TRAJECTORY ERRORS

With this lecture we shall begin the discussion of the effects of the geodetic errors. I begin by considering the station location errors. In the first lecture, we considered the Earth fixed cartesian coordinates of the tracking station. Let its corresponding spherical coordinates be:

$$\begin{aligned}\varphi_R &= \text{geocentric latitude,} \\ &= \sin^{-1} \frac{Z_R}{r_R} ; \\ \lambda_R &= \text{geocentric longitude,} \\ &= \tan^{-1} Y_R/X_R ; \\ r_R &= \text{geocentric radius,} \\ &= \sqrt{X_R^2 + Y_R^2 + Z_R^2} .\end{aligned}$$

Let the errors in these coordinates be $\delta\varphi_R$, $\delta\lambda_R$, δr_R respectively. Then, a representation of these errors in distance units to first order in the errors are:

$$\begin{aligned}E_{r_R} &= \delta r_R , \\ E_{\varphi_R} &= r_R \delta\varphi_R , \\ E_{\lambda_R} &= r_R \cos \varphi_R \delta\lambda_R .\end{aligned}$$

I now wish to rotate these errors into the H, L, Z coordinate system defined in Lecture II.

Rotating first about the station radius vector by the pseudo-azimuth, a_z , (Figures 8A and 8B of Lecture II):

$$\begin{aligned} E_{r_R} & \text{ is unchanged,} \\ E_{L_R} &= E_{\phi_R} \sin a_z + E_{\lambda_R} \cos a_z, \\ E_{Z_T}' &= E_{\phi_R} \cos a_z - E_{\lambda_R} \sin a_z. \end{aligned}$$

Where E_{Z_T}' is perpendicular to \vec{r}_R and lies in the H-Z plane and is frequently referred to as the station cross-track error. Making now a rotation about the L-axis by an angle χ (Figure 6 of Lecture II),

$$\begin{aligned} E_{H_R} &= E_{r_R} \cos \chi - E_{Z_R}' \sin \chi, \\ E_{L_R} & \text{ is unchanged,} \\ E_{Z_R} &= E_{r_R} \sin \chi + E_{Z_R}' \cos \chi. \end{aligned}$$

From Figure 6, it can be seen that

$$\begin{aligned} \sin \chi &= \rho_s \cos e, \\ \cos \chi &= \rho_s \sin e + r_{R,s}. \end{aligned}$$

Successive substitutions for $\sin \chi$, $\cos \chi$ and then E_{Z_R}' yields:

$$E_{H_R} = r_{R,s} E_{r_R} + \rho_s [\sin e E_{r_R} - \cos e \cos a_z E_{\phi_R} + \cos e \sin a_z E_{\lambda_R}]$$

$$E_{L_R} = \sin a_z E_{\varphi_R} + E_{\lambda_R} \cos a_z,$$

$$E_{Z_R} = r_{R,s} [\cos a_z E_{\varphi_R} - \sin a_z E_{\lambda_R}] + \rho_s [\cos e E_{r_R} + \sin e \cos a_z E_{\varphi_R} - \sin e \sin a_z E_{\lambda_R}].$$

These are the expressions for the station error which we shall eventually use in computing the effect of station error on tracking data residuals. From here on we shall assume that these errors are scaled by the mean equatorial radius, R_0 .

I now want to direct our attention to the more involved task of obtaining similar expressions for errors in the satellite motion during the time the satellite is above the station's horizon. We assume that the satellite has been tracked such that satellite position errors may be considered only to first order. We denote the coordinates of the satellite by $r_s, \varphi_s, \lambda_s$ in inertial space. These are related to the osculating kepler elements^{6,7} by the relations:

$$r_s = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos(\beta - \omega)} \quad (\text{units of } R_0)$$

$$\sin \varphi_s = \sin i \sin \beta,$$

$$\cos \varphi_s \cos(\lambda_s - \Omega) = \cos i \sin \beta,$$

$$\cos \varphi_s \sin(\lambda_s - \Omega) = \cos \beta,$$

$$\tan(\lambda_s - \Omega) = \cos i \tan \beta,$$

where:

a = semi-major axis (units of R_0),

ϵ = eccentricity,

i = inclination,

ω = argument of perigee

Ω = right ascension (longitude) of ascending node,

M = mean anomaly,

$\dot{M} = n_0$ = anomalistic mean motion,

β = argument of latitude,

$f = \beta - \omega$ = true anomaly,

$\Phi = M + \omega$.

When a change in the geopotential is made of the form:

$$\Delta U = \frac{K}{R_0 r_s} \left\{ \frac{\Delta K}{K} + \sum_{n=2}^{\infty} \frac{1}{r_s^n} [\Delta J_n P_n(\sin \varphi_s) + \sum_{m=1}^{\infty} P_n^m(\sin \varphi_s) (\Delta C_n^m \cos m \lambda_s + \Delta S_n^m \sin m \lambda_s)] \right\}$$

the equations of motion for the changes in the osculating elements to first order are:

$$\delta \dot{a} = \frac{2}{n_0 a} \frac{\partial \Delta U}{\partial \beta} + O(\epsilon),$$

$$\delta \dot{\epsilon} = \frac{1}{n_0 a} \left[\sin(\beta - \omega) \frac{\partial \Delta U}{\partial a} + \frac{2}{a} \cos(\beta - \omega) \frac{\partial \Delta U}{\partial \beta} \right] + O(\epsilon),$$

$$\begin{aligned}\frac{d\delta i}{dt} &= \frac{1}{n_o a^2} \cot \beta \frac{\partial \Delta U}{\partial i} + O(\epsilon), \\ \sin i \, \delta \dot{\Omega} &= \frac{1}{n_o a^2} \frac{\partial \Delta U}{\partial i} + O(\epsilon), \\ \epsilon \delta \dot{\omega} &= \frac{1}{n_o a} \left[-\cos(\beta - \omega) \frac{\partial \Delta U}{\partial a} + \frac{2}{a} \sin(\beta - \omega) \frac{\partial \Delta U}{\partial \beta} \right] = O(\epsilon), \\ \delta \dot{\Phi} &= -\frac{3}{2} \frac{\delta a}{a} n_o - \frac{2}{n_o a} \frac{\partial \Delta U}{\partial a} - \cos i \, \delta \dot{\Omega} + O(\epsilon).\end{aligned}$$

In the above formulas, quantities such as $\frac{\partial \Delta U}{\partial r_s}$ have been approximated by:

$$\frac{\partial \Delta U}{\partial r_s} = \frac{\partial \Delta U}{\partial a} + O(\epsilon),$$

and $\delta \Phi = \delta M + \delta \omega$ has been used to avoid terms $O(1/\epsilon)$.

Integration of the above differential equations of motion with the appropriate boundary conditions will provide one description of the effect of errors in the geopotential on the satellite trajectory. We shall transform these changes in the osculating elements into the H, L, Z coordinate system in order to discuss these effects on the time dependence of the tracking residuals. However, I first want to give two examples of solutions to these equations to provide a better intuitive feel for the kinds of effects that arise from errors in the geopotential.

Let us first consider the effect of changing the boundary conditions. The general solution of these equations of motion can always be considered as being composed of a particular solution of the inhomogeneous equations (including terms explicitly dependent upon ΔU) and a general solution of the homogeneous part of the equations ($\Delta U \equiv 0$).

Considering the solution of the homogeneous equations first, we set $\Delta U \equiv 0$ and obtain the following constants.

- δa_0 = change in semi-major axis,
- $\delta \epsilon_0$ = change in eccentricity,
- δi_0 = change in inclination,
- $\delta \Omega_0$ = change in right ascension of ascending node,
- $\epsilon_0 \delta \omega_0$ = change in argument of perigee,
- δM_0 = change in mean anomaly,

with

$$\delta \Phi_0(t) = \delta M_0 - 3/2 \frac{\delta a_0}{a_0} n_0(t - t_0) + \text{higher orders},$$

t_0 = some epoch, conveniently chosen to be the epoch of the original orbit.

It can be seen that when $\underline{t_0}$ is chosen as the time of the initial orbit epoch the constants δa_0 , $\delta \epsilon_0$, δi_0 , $\delta \Omega_0$, $\delta \omega_0$, and δM_0 can be interpreted as changes to the orbit parameters at the orbit epoch.

The above constants, which arise mathematically from a solution of the homogeneous perturbed equations of motion, are not trivial additions to the perturbed satellite motion from a physical point of view. When an orbit is determined from tracking data using erroneous station locations and satellite forces, the resulting orbit parameters will obviously be in error even if there is zero error in the tracking data itself. Consequently, when considering the effect of geodetic errors on the

satellite motion, account must be taken of the errors in the orbit parameters themselves. The resulting time dependence of the tracking data residuals due directly to errors in the orbit parameters will be derived using the above solution to the homogeneous equations - keeping in mind that they are not arbitrary but a rather complicated implicit functional of the geodetic errors and amount and distribution of tracking data along the satellite trajectory.

I shall choose one other (relatively simple) example to aid in understanding intuitively the effect of satellite force errors on the satellite motion and eventually on the tracking data residuals. This example allows only an error in the value of J_3 , the so-called pear-shaped term. A particular solution of the above equations of motion for $\Delta J_3 \neq 0$ is to first order. ($\frac{\Delta J_3}{J_2}$ is always considered of first order, ΔJ_3 of second order.)

δa = second order

$$\delta e = 1/2 \frac{\Delta J_3}{J_2} \frac{\sin i}{a} \sin \omega + O(e \frac{\Delta J_3}{J_2}),$$

$$\delta i = O(e \frac{\Delta J_3}{J_2}),$$

$$e \delta \omega = 1/2 \frac{\Delta J_3}{J_2} \frac{\sin i}{a} \cos \omega + O(e^2 \frac{\Delta J_3}{J_2}),$$

$$\delta \Omega = O(e \frac{\Delta J_3}{J_2}),$$

$$\delta \Phi(t) = O(e \frac{\Delta J_3}{J_2}).$$

From these equations it can be seen that an error in J_3 gives rise to periodic errors in the eccentricity and argument of perigee the period being the time of one revolution of perigee.

The example of an errored J_3 is directly generalizable to the form of the errors in the satellite motion arising from errors in the odd zonal harmonics ($\Delta J_n \neq 0$, n odd). Without further remarks, the principal effect of geopotential errors are:⁴

1. Error in even zonal coefficients ($\Delta J_n \neq 0$, n even):
 - a. Secular errors in ω , Ω , Φ (increase approximately linear with time)
 - b. Long period errors in ω , Φ .
 - c. Short (orbital) period errors in all osculating elements.
2. Error in odd zonal coefficients ($\Delta J_n \neq 0$, n odd):

Long period errors in e and ω .
3. Errors in the non-zonal coefficients ($\Delta C_n^m, \Delta S_n^m \neq 0$)

Periodic errors of angular frequency.

$$\omega_m = m (\omega_E - \dot{\Omega}), \quad 1 < m \leq n.$$

As a first step in obtaining the errors in the satellite motion in the H, L, Z system, I shall transform the errors to a moving coordinate system which will also display more clearly the nature of the errors.

This coordinate system is shown in Figure 9A, where:

$$\begin{aligned} \delta r_s(t) &= \text{error in satellite radius (satellite altitude error),} \\ \delta l_s(t) &= \text{error in orbital plane normal to } r_s \text{ (satellite along-track error),} \end{aligned}$$

$\delta Z_s(t)$ = error in direction of satellite angular momentum vector
(satellite cross-track error).

From Figures 9A and 9B it can be seen that

$$\delta \ell_s = r_s [\cos \varphi_s \cos I \delta \Omega + \delta \beta] + \text{second order},$$

$$\delta Z_s = - r_s [\cos \varphi_s \sin I \delta \Omega - \sin \beta \delta i] + \text{second order},$$

Noting from these figures that the local inclination, I , obeys the relations:

$$\cos \varphi_s \cos I = \cos i,$$

$$\cos \varphi_s \sin I = \sin i \sin \beta;$$

$$\delta \ell_s = r_s [\delta \beta + \cos i \delta \Omega] + \text{second order},$$

$$\delta Z_s = r_s [\sin \beta \delta i - \cos \beta \sin i \delta \Omega] + \text{second order}.$$

Using now the relations between the various kepler elements:

$$\delta \beta(t) = \delta f(t) + \delta \omega(t)$$

$$= \delta \Phi(t) + 2[\delta \epsilon \sin(\beta - \omega) - (\epsilon \delta \omega) \cos(\beta - \omega)] + O(\epsilon),$$

and

$$\begin{aligned}\delta r_s &= \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos(\beta - \omega)}, \\ &= \delta a - a[\delta \epsilon \cos(\beta - \omega) - (\epsilon \delta \omega) \sin(\beta - \omega)] + O(\epsilon),\end{aligned}$$

we have:

$$\begin{aligned}\delta r_s(t) &= \delta a - a[\delta \epsilon \cos(\beta - \omega) + (\epsilon \delta \omega) \sin(\beta - \omega)] + O(\epsilon), \\ \delta \ell_s(t) &= a[\delta \Phi + 2 \delta \epsilon \sin(\beta - \omega) - 2(\epsilon \delta \omega) \cos(\beta - \omega)] + O(\epsilon), \\ \delta Z_s(t) &= a[\delta i \sin \beta - \delta \Omega \sin i \cos \beta] + O(\epsilon).\end{aligned}$$

I shall begin the next lecture by discussing the above two examples in the $\delta r_s, \delta \ell_s, \delta Z_s$ system.

Suggested references for Lecture III.

6. Brower, D. and Clemence, G. "Methods of Celestial Mechanics", Academic Press, (1961).
7. Plummer, H. C., "An Introductory Treatise on Dynamical Astronomy", Dover Publications, New York, (1960 edition).

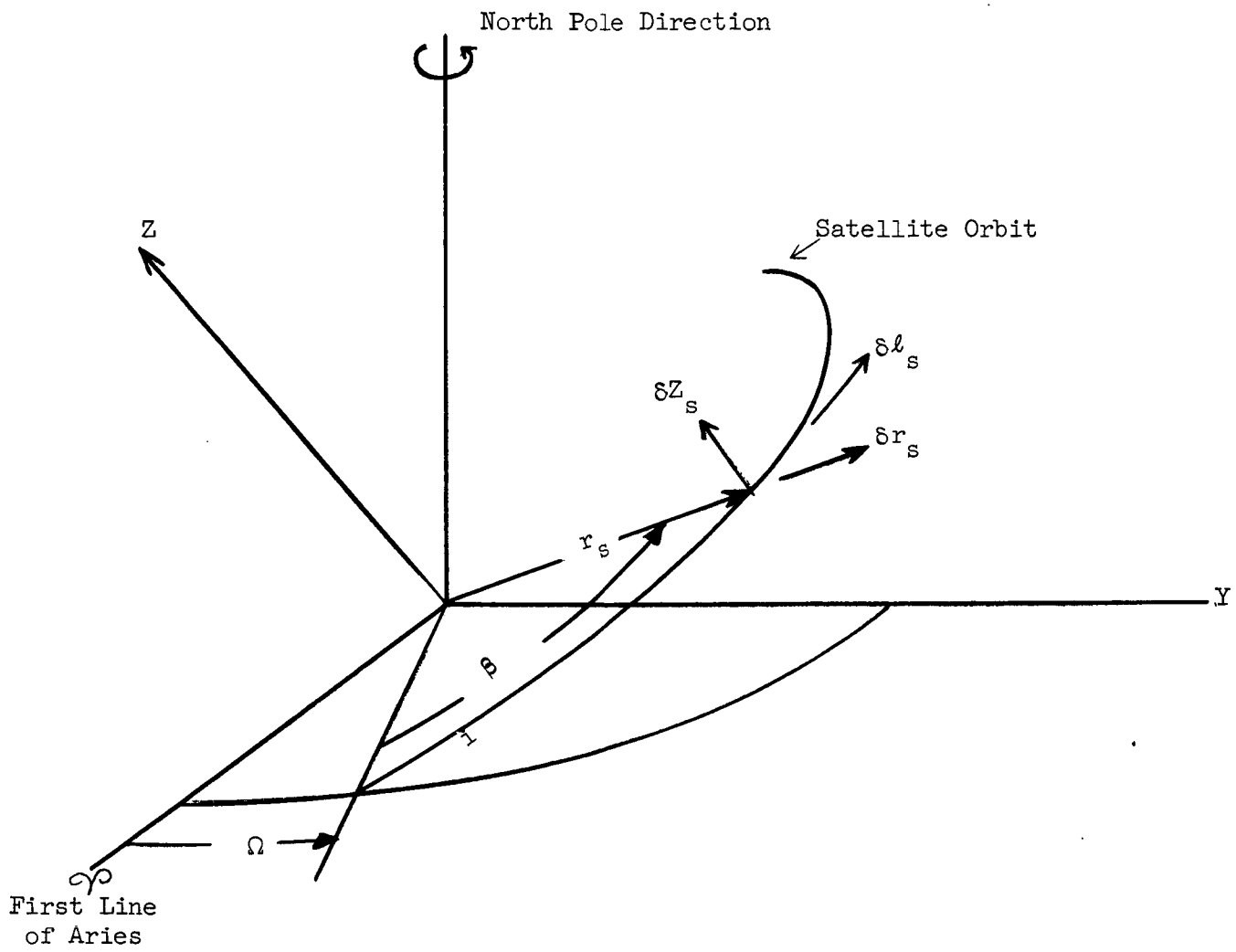


Figure 9A

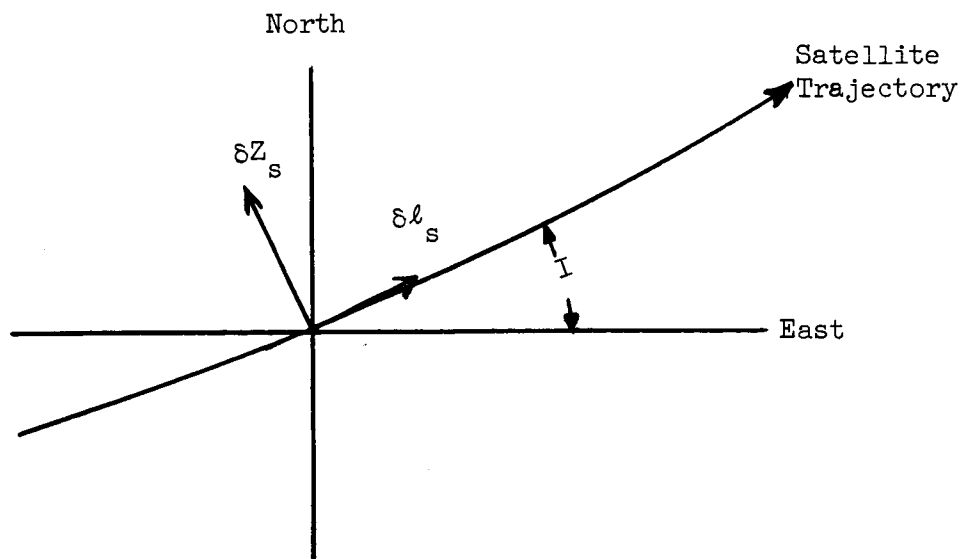


Figure 9B

LECTURE IV

ERROR IN SATELLITE MOTION (Cont'd.)

At the close of the previous lecture we obtained the general expressions for the satellite motion errors in the coordinates δr_s , $\delta \ell_s$, δZ_s given errors in the osculating elements of the orbit. I now wish to consider our two examples in this coordinate space.

1. Errors in the orbit parameters at epoch.

The constant orbit parameter errors can be directly substituted into the expressions for the satellite altitude, along-track and cross-track errors. We then have:

$$\delta r_s(t) = \delta a_o - a[\delta \epsilon_o \cos(\beta - \omega) + (\epsilon_o \delta \omega) \sin(\beta - \omega)] + O(\epsilon),$$

$$\delta \ell_s(t) = a[\delta M_o + \delta \omega_o + \cos i \delta \Omega_o - \frac{3}{2} \frac{\delta a_o}{a} n_o(t - t_o)]$$

$$+ 2(\delta \epsilon_o \sin(\beta - \omega) - (\epsilon_o \delta \omega_o) \cos(\beta - \omega))] + O(\epsilon),$$

$$\delta Z_s(t) = a[\delta i_o \sin \beta - \sin i \delta \Omega_o \cos \beta] + O(\epsilon).$$

Recognizing that the argument of perigee, ω , is a slowly varying function of time, the above expressions can be rewritten in a more transparent form by letting

$$\delta A_0(t) = -a[\delta \epsilon_0 \cos \omega(t) - (\epsilon_0 \delta \omega_0) \sin \omega(t)],$$

$$\delta B_0(t) = -a[\delta \epsilon_0 \sin \omega(t) + (\epsilon_0 \delta \omega_0) \cos \omega(t)],$$

$$\delta l_0 = a[\delta M_0 + \delta \omega_0 + \delta \Omega_0 \cos i],$$

$$\delta l_1 = -\frac{3}{2} \delta a_0,$$

$$\delta l_2 = 2 \delta B_0(t),$$

$$\delta l_3 = -2 \delta A_0(t),$$

$$\delta Z_1 = -a \sin i \delta \Omega_0,$$

$$\delta Z_2 = a \delta i_0,$$

so that when errors exist only in the orbit parameters

$$\delta r_s(t) = -\frac{2}{3} \delta l_1 - \frac{\delta l_3}{2} \cos \beta + \frac{\delta l_2}{2} \sin \beta + O(\epsilon),$$

$$\delta l_s(t) = \delta l_0 + \delta l_1(\beta - \beta_0) + \delta l_2 \cos \beta + \delta l_3 \sin \beta + O(\epsilon),$$

$$\delta Z_s(t) = \delta Z_1 \cos \beta + \delta Z_2 \sin \beta + O(\epsilon).$$

The above equations display the principal time dependence of the errors in the satellite motion when errors exist only in the orbit parameters at the orbit epoch. However, do not overlook the slow time dependence occurring through the motion of perigee and therefore $\delta \ell_2$ and $\delta \ell_3$, and the small time dependence occurring due to the use of the osculating elements for \underline{a} and \underline{i} . As is to be expected, if there is an error in the period of the satellite motion, the satellite along-track error grows linearly with time and the satellite altitude exhibits an altitude error δa_0 which will not average to zero. $\delta \ell_0$ is the position error in the along-track direction at the epoch. It can be seen that the remaining terms in the error equations are oscillatory at the orbital period.

2. Error in the third zonal coefficient, J_3 .

Substituting the errors for the kepler elements corresponding to ΔJ_3 into the expressions for δr_s , $\delta \ell_s$, δZ_s we have:

$$\begin{aligned} \delta r_s(t) &= -a \left[\frac{1}{2} \frac{\Delta J_3}{J_2} \frac{\sin i}{a} \sin \omega \cos(\beta - \omega) + \frac{1}{2} \frac{\Delta J_3}{J_2} \frac{\sin i}{a} \cos \omega \sin(\beta - \omega) \right] \\ &\quad + O\left(\epsilon \frac{\Delta J_3}{J_2}\right) + O(\Delta J_3) \\ &= \frac{1}{2} \frac{\Delta J_3}{J_2} \sin i \sin \beta + O\left(\epsilon \frac{\Delta J_3}{J_2}\right) + O(\Delta J_3) \\ \delta \ell_s(t) &= -\frac{\Delta J_3}{J_2} \sin i \cos \beta + O\left(\epsilon \frac{\Delta J_3}{J_2}\right) + O(\Delta J_3) \\ \delta Z_s(t) &= O\left(\epsilon \frac{\Delta J_3}{J_2}\right) + O(\Delta J_3). \end{aligned}$$

A very interesting point can be seen from these equations. We had previously noted that the errors in the kepler elements due to an error in J_3 were long period to first order - that is order $\frac{\Delta J_3}{J_2}$. However, once transformed to a coordinate system that is closer to giving a direct measure of the satellite position error, the effects (to this same order) become short period. Because the dominant effect is now short period the resulting satellite errors exhibit a similar time dependence to the errors caused by orbit parameter errors along (example 1). This means that over short intervals of time, say a few days it is possible to "soak up" most of the error due to this geopotential error by appropriate adjustment of the satellite orbit parameters.

To exhibit this effect clearly, we combine the two previous examples assuming that no errors exist except in the value for J_3 and allow an error in the orbit parameters which will minimize the effect of J_3 being in error. From the previous results, we have:

$$\begin{aligned}\delta A(t) &= \delta A_0(t) \\ &= -a[\delta \epsilon_0 \cos \omega(t) - (\epsilon_0 \delta \omega_0) \sin \omega(t)], \\ \delta B(t) &= \delta B_0(t) - a \frac{\Delta J_3}{2J_2} \frac{\sin i}{a} \\ &= -a \left[\frac{\Delta J_3}{2J_2} \frac{\sin i}{a} + \delta \epsilon_0 \sin \omega(t) + (\epsilon_0 \delta \omega_0) \cos \omega(t) \right], \\ \delta \ell_0 &= a[\delta M_0 + \delta \omega_0 + \cos i \delta \Omega_0],\end{aligned}$$

$$\delta \ell_1 = - 3/2 \delta a_0,$$

$$\delta L_2(t) = 2 \delta B(t) = \delta \ell_2(t) - a \frac{\Delta J_3}{J_2} \frac{\sin i}{a},$$

$$\delta L_3(t) = - 2 \delta A(t) = \delta \ell_3(t),$$

$$\delta Z_1 = - a \sin i \delta \Omega_0,$$

$$\delta Z_2 = a \delta i_0,$$

and:

$$\delta r_s(t) = - 2/3 \ell_1 - \frac{\delta \ell_3(t)}{2} \cos \beta + \frac{\delta L_2(t)}{2} \sin \beta + \text{higher orders},$$

$$\delta \ell_s(t) = \delta \ell_0 + \delta \ell_1(\beta - \beta_0) + \delta L_2(t) \cos \beta + \delta \ell_3(t) \sin \beta + \text{higher orders},$$

$$\delta Z_s(t) = \delta Z_1 \cos \beta + \delta Z_2 \sin \beta + \text{higher orders}.$$

These equations have intentionally been written to look formally like those which represented only orbit parameter errors. The only difference that occurs when ΔJ_3 is not zero to the order considered here is:

$$\delta L_2(t) - \delta \ell_2(t) = - \frac{\Delta J_3}{J_2} \sin i + \text{higher orders}.$$

Since $\delta \ell_2(t)$, and therefore $\delta L_2(t)$, are varying with time very slowly, it becomes difficult to separate an orbit parameter error from this type of

geodetic error. This tendency for orbit parameter adjustment to hide geodetic errors, exhibited in this example, is a general result for many types of geodetic errors, particular errors in the zonal harmonic coefficients of the geopotential. It is for this reason that long satellite trajectories are usually required to determine accurately the zonal harmonic coefficients in the presence of other errors such as station location errors and experimental data errors.^{8,9,10}

We have considered the general character of the errors in the satellite motion over long spans of time through two examples. I now want to consider in more detail the effect of these errors on the tracking data for a specific pass of the satellite above a specific station's horizon. To do this we transform the satellite motion errors to the H, L, Z coordinate system. For some given pass, the H-axis passes through the satellite position at closest approach and is fixed in inertial space. Figure 10 gives the geometry of the errors in the δr_s , δl_s moving coordinate system relative to the fixed coordinate system of H and L. From Figure 10; it can be seen that:

$$\delta H_s = \delta r_s \cos \Delta\beta - \delta l_s \sin \Delta\beta,$$

$$\delta L_s = \delta r_s \sin \Delta\beta + \delta l_s \cos \Delta\beta,$$

$$\delta Z_s \text{ unchanged}$$

$$\Delta\beta = \beta(t) - \beta(t_c),$$

$$t_c = \text{time of closest approach.}$$

Letting

$$C(\Delta\beta) = 1 - \cos \Delta\beta,$$

it can be seen that during the pass $|C(t)| \ll 1$ for near-earth satellites.

Rewriting the above equations:

$$\delta H_s = \delta r_s - \delta \ell_s \sin \Delta\beta - \delta r_s C(\Delta\beta),$$

$$\delta L_s = \delta \ell_s + \delta r_s \sin \Delta\beta - \delta \ell_s C(\Delta\beta),$$

$$\delta Z_s \text{ unchanged.}$$

The procedure from here on involves expanding $\delta r_s(t)$, $\delta \ell_s(t)$, and $\delta Z_s(t)$ in the functions $\sin \Delta\beta$, $C(\Delta\beta) = 1 - \cos \Delta\beta$, etc. and then by substitution into the above equations for δH_s and δL_s , express the time dependence of the satellite errors in the H, L, Z coordinate system in functions of the form $\sin \Delta\beta$, $C(\Delta\beta)$, $\sin \Delta\beta C(\Delta\beta)$, etc. This procedure can be done in general but is not too useful to the development of a physical understanding of the effects of the errors. Consequently, I shall make this transformation using the two examples discussed previously, referring you to reference 5 given in Lecture II, for consideration of the general case.

I use a subscript c to denote a time dependent quantity evaluated at $t = t_c$. The result then becomes:

$$\begin{aligned} \delta H_s(\beta_c, \Delta\beta(t)) = & \delta r_c - [\delta \ell_c + \delta A_c \sin \beta_c - \delta B_c \cos \beta_c] \sin \Delta\beta \\ & - [\delta r_c - 3 \delta A_c \cos \beta_c - 3 \delta B_c \sin \beta_c] C(\Delta\beta) \\ & - [\delta A_c \sin \beta_c - \delta B_c \cos \beta_c] \sin \Delta\beta C(\Delta\beta) + O(C^2) + \text{higher orders,} \end{aligned}$$

$$\begin{aligned}\delta L_s(\beta_c, \Delta\beta(t)) &= \delta\ell_c + [\delta r_c - 2\delta A_c \cos \beta_c - 2\delta B_c \sin \beta_c] \sin \Delta\beta - \delta\ell_c C(\Delta\beta) \\ &+ [\delta A_c \cos \beta_c + \delta B_c \sin \beta_c] \sin \Delta\beta C(\Delta\beta) \\ &+ O(C^2) + \text{higher orders},\end{aligned}$$

$$\begin{aligned}\delta Z_s(\beta_c, \Delta\beta(t)) &= \delta Z_c + [\delta\Omega_o \sin i \sin \beta_c + \delta i_o \cos \beta_c] \sin \Delta\beta \\ &- \delta Z_c C(\Delta\beta) + \text{higher orders},\end{aligned}$$

where:

$$\delta r_c = \delta a_o + \delta A_c \cos \beta_c + \delta B_c \sin \beta_c,$$

$$\delta\ell_c = a(t_c)[\delta M_o + \delta\omega_o + \delta\Omega_o \cos i_c] - 2/3 \delta a_o(\beta_c - \beta_o)$$

$$+ 2\delta B_c \cos \beta_c - 2\delta A_c \sin \beta_c,$$

$$\delta Z_c = -\delta\Omega_o \sin i(t_c) \cos \beta_c + \delta i_o \sin \beta_c,$$

$$\delta A_c = -a(t_c)[\delta\epsilon_o \cos \omega(t_c) - (\epsilon_o \delta\omega_o) \sin \omega(t_c)],$$

$$\delta B_c = -a(t_c)\left[\frac{\Delta J_3}{2J_2} \frac{\sin i(t_c)}{a(t_c)} + \delta\epsilon_o \sin \omega(t_c) + (\epsilon_o \delta\omega_o) \cos \omega(t_c)\right].$$

In developing these formulas we have used the relations:

$$\cos (\beta_c + \Delta\beta) = \cos \beta_c - \sin \beta_c \sin \Delta\beta - \cos \beta_c C(\Delta\beta),$$

$$\sin (\beta_c + \Delta\beta) = \sin \beta_c + \cos \beta_c \sin \Delta\beta - \sin \beta_c C(\Delta\beta),$$

$$\sin^2 \Delta\beta = 1 - \cos^2 \Delta\beta = 2 C(\Delta\beta) + O(C^2)$$

and where $-2/3 \delta a_0 \Delta\beta$ has been considered negligible by virtue of our assumption that the orbit has been "tracked" to reasonable accuracy so that $\delta a_0 \beta(t_c)$ is not large.

In the next lecture, we shall use these relations to investigate the residuals for various types of data.

References

8. O'Keefe, J.A., Eckels, A., and Squires, R. K., "The Gravitational Field of the Earth", Astron. J., v.64, p. 245.
9. Newton, R. R., Hopfield, H.S., and Kline, R. C., "Odd Harmonics of the Earth's Gravitational Field", Nature, v.190, p. 617.
10. Kozai, Y., "Numerical Results From Orbits", SAO Special Report No. 101, July 31, 1962.

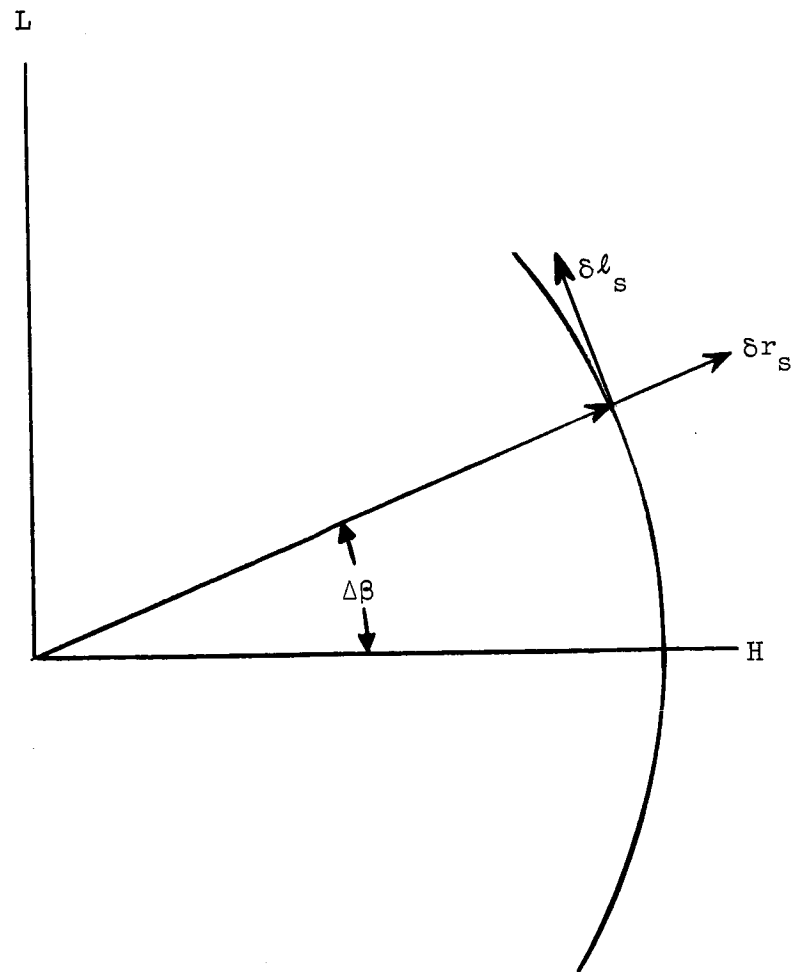


Figure 10

LECTURE V

EFFECT OF GEODETIC ERRORS ON DATA RESIDUALS

For the fifth and final lecture I shall use the previous results to consider the effect of station and geopotential errors on tracking data residuals. By data residuals I mean:

Data Residual =

Experimental data point - Theory at time of data point, where, as stated in Lecture II, we neglect experimental noise and instrumentation contributions to the residuals.

Clearly, the error in the slant range vector is:

$$\vec{\delta\rho} = \vec{\delta r}_s(t) - \vec{\delta r}_R(t),$$

which, in the H,L,Z coordinate system is:

$$\begin{aligned} \vec{\delta\rho} &= \begin{pmatrix} \delta\rho_H \\ \delta\rho_L \\ \delta\rho_C \end{pmatrix}, \\ &= \begin{pmatrix} H_s(\theta_C, \Delta\theta) - E_{H_T} \\ L_s(\theta_C, \Delta\theta) - E_{L_T} \\ Z_s(\theta_C, \Delta\theta) - E_{Z_T} \end{pmatrix} + \text{second order}, \end{aligned}$$

where we discussed the station error E_{H_T} , E_{L_T} , E_{Z_T} in Lecture III and discussed the nature of H_s , L_s , Z_s in Lecture IV.

Corresponding to this error, the error in the scalar slant range, e.g., the slant range data residuals are given by:

$$\begin{aligned}\delta\rho &\equiv |\vec{\rho} + \delta\vec{\rho}| - \rho = \frac{1}{\rho} \vec{\rho} \cdot \delta\vec{\rho} + \text{second order,} \\ &= \hat{\rho} \cdot \delta\vec{\rho} + \text{second order,}\end{aligned}$$

where, from Lecture II:

$$\vec{\rho}(t) = r_s \begin{pmatrix} \rho_s \cos \theta - C(t) \\ \sin \Delta\beta(t) \\ - \rho_s \sin \theta \end{pmatrix} + \text{first order,}$$

$$\rho(t) = r_s [\rho_s^2 + 2 \alpha_s C(t)]^{\frac{1}{2}} + \text{first order,}$$

$$\alpha_s = 1 - \rho_s \cos \theta,$$

$$C(t) = 1 - \cos \Delta\beta(t).$$

The error in the slant range unit vector, e.g., the angular data residuals are give by:

$$\delta\hat{\rho} = \delta\left(\frac{\vec{\rho}}{\rho}\right) = \frac{\delta\vec{\rho}}{\rho} - \hat{\rho} \frac{\delta\rho}{\rho}$$

or; the angular error scaled to distance error is

$$\rho \delta \hat{\rho} = \delta \vec{\rho} - \hat{\rho} (\hat{\rho} \cdot \delta \vec{\rho}).$$

Finally, the error in the scalar slant range rate, e.g., doppler data residuals are given by:

$$\begin{aligned} \frac{d}{dt} \delta \rho &= \frac{d}{dt} \left[\frac{1}{\rho} (\rho \delta \rho) \right] \\ &= \frac{1}{\rho^3} \left[- \frac{1}{2} (\rho \delta \rho) \frac{d\rho^2}{dt} + \rho^2 \frac{d}{dt} (\rho \delta \rho) \right]. \end{aligned}$$

Each of the above types of residuals can be computed by substituting in the appropriate expressions for the error in the vector slant range.

Using now the two examples in Lecture IV as a guide, we can write

$$\delta \vec{\rho}(t) = \delta \vec{\rho}_C + \delta \vec{\rho}_1 \sin \Delta \beta(t) + \delta \vec{\rho}_2 C(t) + \text{higher orders}$$

where

$$\delta \vec{\rho}_C = \delta \vec{r}_S(t_c) - \delta \vec{r}_R(t_c).$$

(The proof of this form for general geopotential errors is lengthy and is given in reference 5 of Lecture II.) Substituting this form into the above expressions for slant range residuals:

- V.4 -

$$\begin{aligned}
 \frac{\rho(t)}{r_s} \delta \rho(t) &= \rho_s [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] \\
 &+ [\delta \rho_{L_C} + \rho_s (\cos \theta \delta \rho_{H_1} - \sin \theta \delta \rho_{Z_1})] \sin \Delta \beta(t) \\
 &+ [2 \delta \rho_{L_1} - \delta \rho_{H_C} + \rho_s (\cos \theta \delta \rho_{H_2} - \sin \theta \delta \rho_{Z_2})] C(t) \\
 &+ O[\sin \Delta \beta C(t)] + \text{higher orders.}
 \end{aligned}$$

For satellites whose altitude is of the order of 1000 km,

$$\rho_s \leq .25$$

$$C(t) \leq .15 .$$

Therefore, to a fair approximation:

A. Scalar slant range residuals:

$$\begin{aligned}
 \frac{\rho(t)}{r_s} \delta \rho(t) &= \rho_s [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] + [\delta \rho_{L_C} + O(\rho_s)] \sin \Delta \beta(t) \\
 &+ [2 \delta \rho_{L_1} - \delta \rho_{H_C} + O(\rho_s)] C(\Delta \beta) + O[\sin \Delta \beta \cdot C(t)].
 \end{aligned}$$

Similarly, by substitution into the expression the other types of data:

B. Angular Residuals:

$$\left(\frac{\rho(t)}{r_s}\right)^3 r_s \delta \hat{\rho} = \delta \hat{\rho}_C + \delta \hat{\rho}_1 \sin \Delta \beta(t) + \delta \hat{\rho}_2 c(t) + O[\sin \Delta \beta \cdot c(t)],$$

$$\delta \hat{\rho}_C = \rho_s^2 \begin{pmatrix} \delta \rho_{H_C} - \cos \theta [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] \\ \delta \rho_{L_C} \\ \delta \rho_{Z_C} + \sin \theta [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] \end{pmatrix},$$

$$\delta \hat{\rho}_1 = \rho_s \begin{pmatrix} -\cos \theta \delta \rho_{L_C} + O(\rho_s) \\ -[\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] + O(\rho_s) \\ \sin \theta \delta \rho_{L_C} + O(\rho_s) \end{pmatrix},$$

$$\delta \hat{\rho}_2 = \begin{pmatrix} 2 \delta \rho_{H_C} + O(\rho_s) \\ O(\rho_s) \\ 2 \delta \rho_{Z_C} + O(\rho_s) \end{pmatrix}.$$

C. Range Rate Residuals:

$$- \frac{1}{n_o} \left(\frac{\rho(t)}{r_s} \right)^3 \frac{d}{dt} \delta \rho(t) = \rho_s^2 [\delta \rho_{L_C} + O(\rho_s)]$$

$$- \rho_s [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C} + O(\rho_s)] \sin \Delta \beta(t) + O(\rho_s^2 C(t)),$$

where $n_o = \dot{\beta}(t_c)$.

These results are summarized in the following table for purposes of comparison, where they have been scaled to like functions of time. It should be noted that in the above expressions and the following table the angular residuals have been written as a three-dimensional vector in the H, L, Z coordinate system. However, in reality, the residuals are only a two dimensional vector since

$$\vec{\rho}(t) \cdot \hat{\delta \rho}(t) = 0.$$

This table summarizes the largest contributions to the expressions for data residuals when experimental errors are neglected. Clearly, the errors $\delta \rho_{H_C}$, $\delta \rho_{L_C}$, and $\delta \rho_{Z_C}$, can be expressed in terms of the station location errors, orbit parameters errors, and geopotential errors following the procedure outlined in Lectures III and IV. A rough sketch of the time dependence of the various terms are given in Figures 11 so that for any given geodetic error its effect on the time dependence of the data residuals can be found.

Several interesting conclusions can be drawn from this table. First, it can be seen that for comparable signal to noise ratios, range and range rate data yield roughly the same information. This, at first

SUMMARY OF RANGE, RANGE RATE, AND ANGLE DATA RESIDUALS

TO $O(\rho_s C(t))$

TYPE OF DATA TIME DEPENDENCE	RANGE $\delta \rho(t)$	RANGE RATE $-\frac{1}{n_o} \frac{d}{dt} \delta \rho(t)$	ANGLE $r_s \delta \rho(t) = r_s \begin{pmatrix} \delta \rho_H(t) \\ \delta \rho_L(t) \\ \delta \rho_Z(t) \end{pmatrix}$
$\left(\frac{r_s}{\rho(t)}\right)^3 \rho_s^3$ (symmetric in time)	$\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}$	$\frac{\delta \rho_{L_C}}{\rho_s} + O(1)$	$\frac{1}{\rho_s} \begin{pmatrix} \rho_{H_C} - \cos \theta [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] \\ \delta \rho_{L_C} \\ \delta \rho_{Z_C} + \sin \theta [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] \end{pmatrix}$
$\left(\frac{r_s}{\rho(t)}\right)^3 \rho_s^2 \sin \Delta \beta(t)$ (anti-symmetric in time)	$\delta \rho_{L_C} + O(\rho_s)$	$-\frac{1}{\rho_s} [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] + O(1)$	$\frac{1}{\rho_s} \begin{pmatrix} -\cos \theta \delta \rho_{L_C} + O(\rho_s) \\ -[\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] + O(\rho_s) \\ \sin \theta \delta \rho_{L_C} + O(\rho_s) \end{pmatrix}$
$\left(\frac{r_s}{\rho(t)}\right)^3 \rho_s C(t)$ (symmetric in time)	$2[\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] + O(\rho_s)$	$O(\rho_s)$	$\frac{2}{\rho_s} \begin{pmatrix} \delta \rho_{H_C} + O(\rho_s) \\ O(\rho_s) \\ \delta \rho_{Z_C} + O(\rho_s) \end{pmatrix}$

$$\frac{r_s^2}{\rho(t)} = \frac{1}{\rho_s^2 + 2 \alpha_s C(t)}, \quad C(t) = 1 - \cos \Delta \beta(t), \quad \alpha_s = 1 - \rho_s \cos \theta$$

n_o = satellite mean motion.

glance, is surprising since one would suspect that range rate data, being the time derivative of the range, would lose some information (roughly analogous to the constant of integration if one attempted to integrate the range rate data to obtain range). Clearly, this is not true except to note that it has been assumed that the transmitter frequency of the satellite which generates the doppler data is known exactly so that the incoming signal can "zero-beat" out the satellite transmitter frequency. (To the extent that this is not true, a term which is constant with time should be added which can easily be separated out from the time dependence noted in the table). The second conclusion is that when range and/or range rate is measured, the following measurements of the relative error between satellite and station can be made from a single pass.

$$\delta \rho_{LC},$$

$$\delta \rho_{HC} \cos \theta - \sin \theta \delta \rho_{ZC}.$$

Considering now the parameters that can be determined with angular residuals from a single pass, we have:

$$\delta \rho_{LC},$$

$$\begin{aligned} \rho_{HC} &= \cos \theta [\cos \theta \delta \rho_{HC} - \sin \theta \delta \rho_{ZC}] \\ &= \sin \theta [\sin \theta \delta \rho_{HC} + \cos \theta \delta \rho_{ZC}], \end{aligned}$$

$$\begin{aligned} \delta \rho_{Z_C} + \sin \theta [\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}] \\ = \cos \theta [\sin \theta \delta \rho_{H_C} + \cos \theta \delta \rho_{Z_C}], \end{aligned}$$

and

$$[\cos \theta \delta \rho_{H_C} - \sin \theta \delta \rho_{Z_C}],$$

so that more information is available in optical data than range or range rate data for equivalent signal to noise ratios and data rates.

Touching, for the moment on the relative merits of different types of data, the following should be noted. Range and range rate systems are usually radio tracking systems and consequently have all weather capabilities and designed to yield very high data rates. I believe most people agree that no radio tracking system significantly exceeds the data point accuracy of a good optical (angle) tracking instrument. However, optical tracking systems are not all weather and as a maximum can only take data at night. Including the tedious job of reducing the optical photographs, we can see that range and range rate systems yield high data rates in all weather conditions but per satellite pass may yield less information than a high quality set of optical data. Consequently, it would appear that a high quality radio range or range rate system and a high quality optical tracking system are complimentary to each other. For example, optical data provides an excellent means for monitoring the accuracy of radio tracking systems. This fact has been recognized in the ANNA geodetic satellite^{11,12} in

which was flown an active flashing light to aid in obtaining increased optical data rates together with instrumentation for two radio tracking systems.

So far we have been concerned only with the data residuals for a single satellite pass. Clearly, when considering many such sets of data residuals, one has the capability of measuring the time dependence of the orbit error over long time spans to gain information on geopotential terms which produce secular and long-period effects. When using such data to make a significant improvement in current values for station position parameters and coefficients of the expansion of the geopotential, a sufficiently large number of parameters must be inferred from the data that it is essential to have very large amounts of tracking data. In fact experience has shown that one really needs many satellites at differing inclinations, to accurately determine the non-zonal coefficients of the geopotential.

The techniques and associated computer programs which are used to perform such determinations of geodetic parameters are outside the scope of this series of lectures. It is sufficient to note that one must have available high quality tracking data from many satellites and extensive computer programs before such an attempt is capable of improving on current accuracies. I hope that this series of lectures has clarified some of the problems involved in the design of such programs and highlighted the essentials of the information content of the data residuals that would be used.

11. Macomber, M. M., "Project ANNA", Proceedings of the First International Symposium on the Use of Artificial Satellites for Geodesy, Washington, D. C., 1962.
12. Guier, W. H., "Navigation Using Artificial Satellites - The Transit System", Proceedings of the First International Symposium on the Use of Artificial Satellites for Geodesy, Washington, D. C., 1962.

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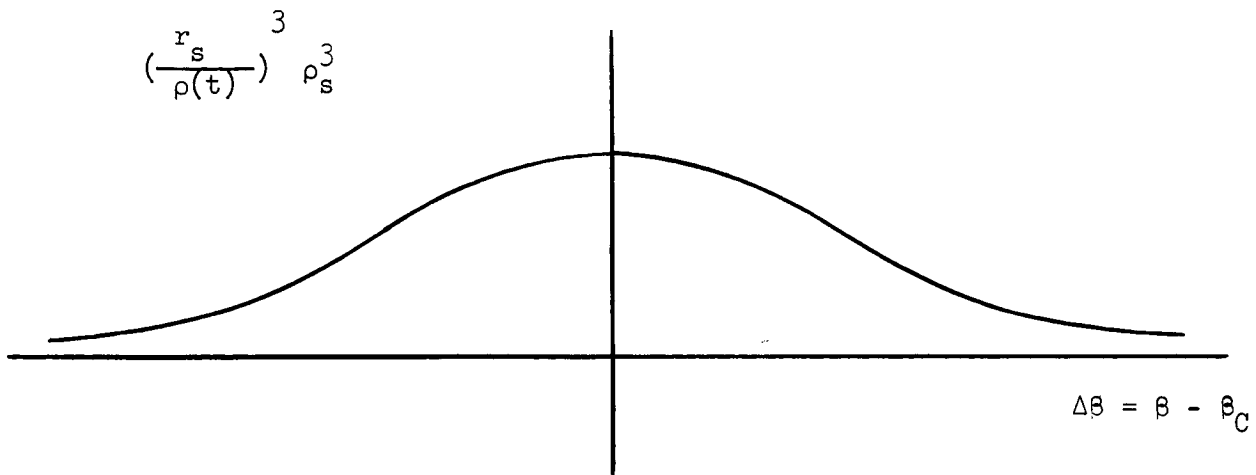


FIGURE 11A

FIRST SYMMETRIC TIME DEPENDENCE

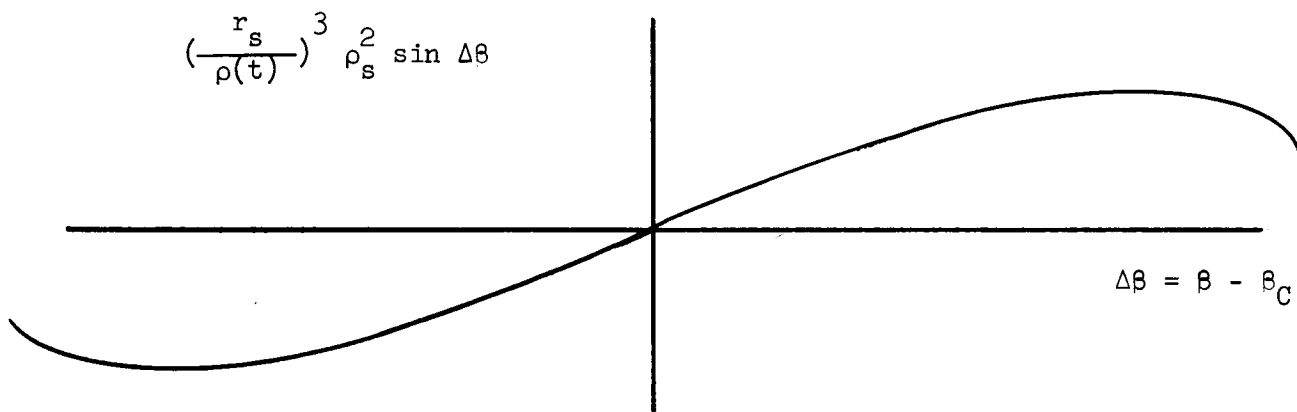


FIGURE 11B

FIRST ANTI-SYMMETRIC TIME DEPENDENCE

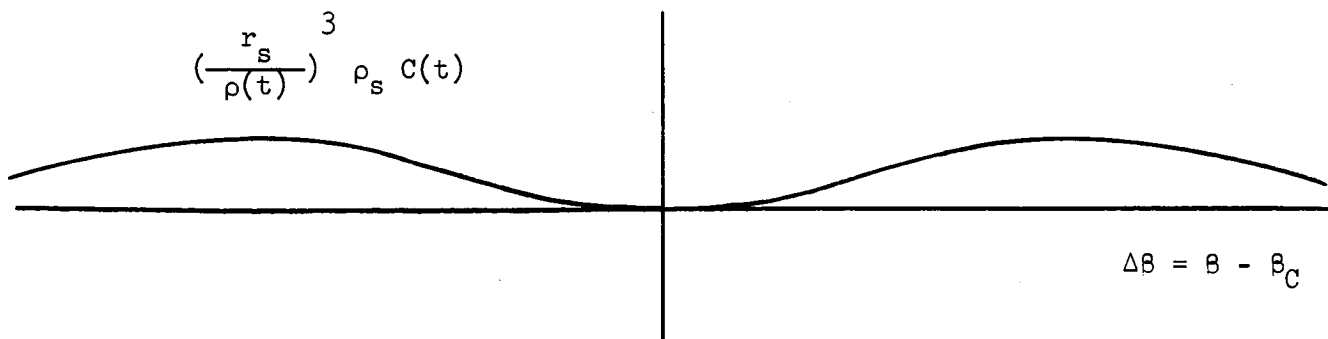


FIGURE 11C

SECOND SYMMETRIC TIME DEPENDENCE

Geodetic Problems and Satellite Orbits

by

Dr. William H. Guier

Geodetic Problems and Satellite Orbits

The main problems to be discussed in this series of lectures will be:

1. Methods of finding and specifying where tracking stations are on the surface of the earth. The location of tracking stations being with respect to the spin axis and center of gravity of the earth.
2. Discussion of satellite motion in the earth's force field. In particular the effect of the various harmonic terms in the earth's potential on the motion of a satellite.

Some standard references are:

1. Bomford, B.G., "Geodesy," Clarendon Press (1952).
2. Heiskanen, W.A. and Vening Meinesz, F.A., "The Earth and Its Gravity Field," McGraw-Hill (1958).

Earth's Potential

Assuming that the earth's force field is $+\text{grad}U$, we have:

$$U(R, \phi, \lambda) = \frac{K}{R} \left\{ 1 + \sum_{n=2}^{\infty} J_n \left(\frac{R_0}{R} \right)^n P_n(\sin \phi) + \sum_{n=2}^{\infty} \sum_{m=1}^n \left(\frac{R_0}{R} \right)^n P_n^m(\sin \phi) \left[C_n^m \cos n\lambda + S_n^m \sin n\lambda \right] \right\}$$

for the earth's potential, where R is distance measured from the center of the earth, ϕ is latitude, λ is longitude. $P_n(\sin \phi)$ and $P_n^m(\sin \phi)$ are the standard and associated Legendre polynomials. R_0 is the average earth's radius, while K , J_n , C_n^m , S_n^m are constants to be determined.

If we make the logical assumptions that our coordinate system has its origin at the center of the earth, and that ϕ is measured

relative to the earth's spin axis (which we assume goes through the center of gravity of the earth) and both the center of gravity of the earth and the earth's spin axis are fixed relative to the earth's crust, it follows that $J_1 = C'_1 = S'_1 = 0$ and $C'_2 = S'_2 = 0$.

We will not be interested in the gravitational field inside the earth, and will assume that the gravitational field is time independent.

Satellite Orbits

Beside considering the equations of motion of a satellite, we will also consider orbit parameters.

There are many restrictions we will make in our study of the motion of a satellite:

1. The equations of motion will be non-relativistic.
2. Air-drag and electromagnetic forces will not be considered.

We will assume that the satellite is above 1000 kilometers where air-drag is negligible.

3. We will assume a low eccentricity $e \leq .05$.
4. We will neglect all errors in data. This will include such errors as:
 - a. Radio data, i.e., ionosphere and troposphere refraction will be neglected.
 - b. "Front End" receiver noise, that is, detector noise will be neglected.
 - c. Data goofs such as bad card punches and systematic errors will be neglected.
5. Extra-terrestrial forces will be neglected, i.e., the force of the sun, moon and planets.

We will assume some force field for the earth, and some numerical way of solving the equations of motion. At a particular time t_k we will have computed and measured orbital parameters. The difference between the measured and predicted values will be called the data residuals.

In equation form:

$$R(t_k) = T(t_k \mid \text{Forces, station location, orbit parameters, methods of computation}) - E(\text{experimental error}).$$

We will try to minimize the residual errors in the least square sense.

Kinds of Data

The kinds of data we will be concerned with are:

1. Optical data giving the slant range unit vector.
2. Radar range data giving the scalar slant range.
3. Range rate data from Doppler measurements.

Naturally some stations will have mixtures of these three types of data. These lectures will stress type (3).

Finding Station Locations

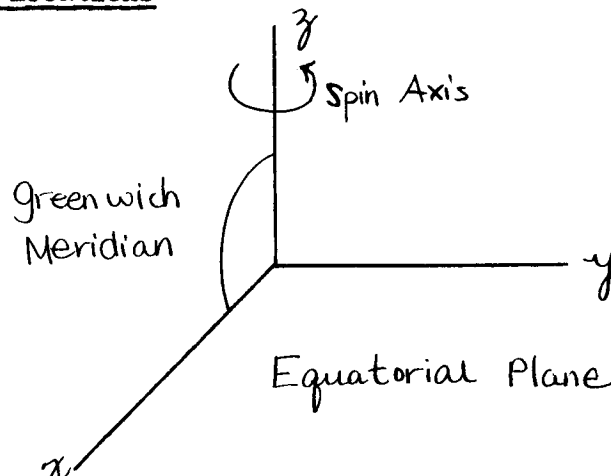


Figure 1.

We will now develop a method for specifying where a tracking station (x_T, y_T, z_T) is with respect to the coordinate system shown in Figure 1.

We first approximate the earth by a spheroid or oblate ellipsoid of revolution. Specifying a spheroid is referred to as a datum. Thus the NASA world datum is

$$R_0 = 6378.166 \text{ kilometers} \quad f_0 = 1/298.24$$

where R_0 is mean equatorial radius of the world scaling factor and f , the so-called flattening, is related to eccentricity by the formula:

$$f = 1 - \sqrt{1 - e^2}.$$

With this scaling factor the earth's semi-major axis is 1 in the NASA world datum.

Geodetic Latitude

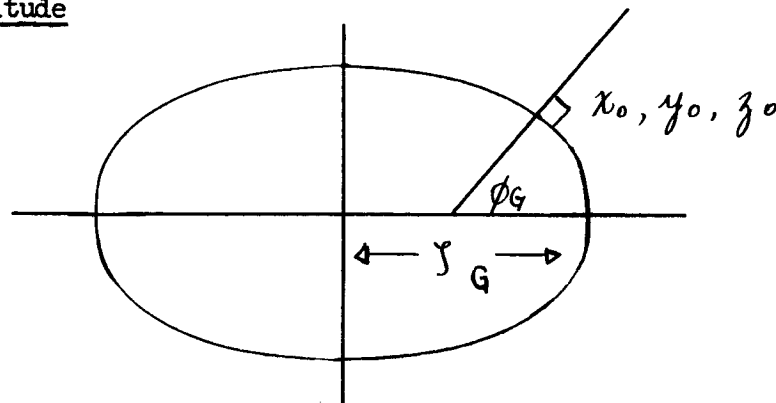


Figure 2.

Positions on a spheroid are given by reference to geodetic latitude, defined in Figure 2. (One uses the normal to the surface since this is the approximate way a weight would drop.) Some

standard formulas are:

$$\rho_G = a / \sqrt{1 + (1 - f)^2 \tan^2 \phi_G}$$

$$z_0 = (1 - f)^2 \rho_G \tan \phi_G$$

$$\frac{\rho_G^2}{a^2} + \frac{z_0^2}{a^2(1 - f)^2} = 1$$

$$\text{Geodetic longitude} = \tan^{-1} \frac{y_0}{x_0}$$

Equipotential Surface

An equipotential surface Ψ on the surface of the earth, taking into account the earth's rotation is:

$$\Psi = U + \frac{W_E^2}{2} (X^2 + Y^2) = \text{Constant.}$$

To terms of higher order, one has:

$$U_0 = \frac{K}{R} \left[1 + \frac{J_2}{2} \left(\frac{R_0}{R} \right)^2 \left(3 \frac{Z^2}{R^2} - 1 \right) \right]$$

$$\text{Setting } x_0 = \frac{X_0}{R_0}, \quad y_0 = \frac{Y_0}{R_0}, \quad z_0 = \frac{Z_0}{R_0}, \quad x_0^2 + y_0^2 = \rho_G^2$$

on the surface, one has:

$$\Psi_0 = U_0 + \frac{W_E^2}{2} R_0^2 \rho_G^2.$$

Two formulas are:

$$\frac{\Psi_{0R_0}}{K} \approx 1 - \frac{J_2}{2} + \frac{W_E^2 R_0^3}{2K}$$

$$f \approx -\frac{3}{2} J_2 + \frac{W_E^2 R_0^3}{2K}$$

Geoid

A geoid is an equipotential surface that best approximates mean sea-level.

A co-geoid is the best approximation to a geoid over land masses where the geoid can't be actually measured.

To give position on a geoid, one uses the geoidal height H which is the distance between co-geoid and spheroid measured along the normal to the spheroid. Thus position on the co-geoid is given by the formulas:

$$x_G = (\rho_G + H \cos \phi_G) \cos \lambda_G$$

$$y_G = (\rho_G + H \cos \phi_G) \sin \lambda_G$$

$$z_G = z_0 + H \sin \phi_G$$

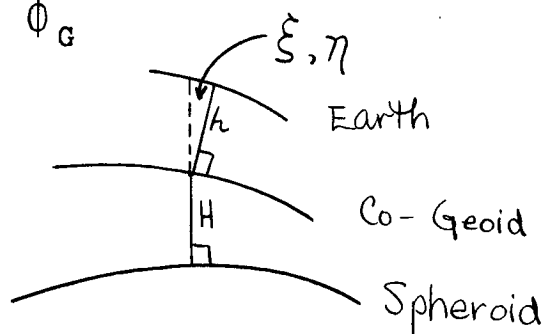


Figure 3.

To finally specify position on earth, one must have three other parameters.

h = elevation above co-geoid (measured perpendicular to co-geoid)

Since the perpendicular to the co-geoid may not be parallel to the perpendicular to the spheroid, we need:

ξ = deflection in meridian between perpendicular to co-geoid and perpendicular to spheroid

η = deflection in prime vertical between perpendicular to co-geoid and perpendicular to spheroid.

The final formulas for the position of a tracking station are:

$$X_T = [\rho_G + (H + \eta) \cos \phi_G] \cos \lambda_G - h [\xi \sin \phi_G \cos \lambda_G + \eta \cos \phi_G \sin \lambda_G] + \delta_x + \text{terms of higher order}$$

$$Y_T = [\rho_G + (H + \eta) \cos \phi_G] \sin \lambda_G - h [\xi \sin \phi_G \sin \lambda_G - \eta \cos \phi_G \cos \lambda_G] + \delta_y + \text{terms of higher order}$$

$$Z_T = Z_0 + (H + \eta) \sin \phi_G + \xi h \cos \phi_G + \delta_z + \text{terms of higher order}$$

The three terms δ_x , δ_y , δ_z have been added to account for changing from one datum to another. For example, from North American datum to NASA world datum.

Letting the coordinates of a station be x_R, y_R, z_R then

$$r_R = \sqrt{x_R^2 + y_R^2 + z_R^2}$$

$$\phi_R = \text{geocentric latitude} = \sin^{-1} z_R / r_R$$

$$\lambda_R = \text{geocentric longitude} = \tan^{-1} y_R / x_R$$

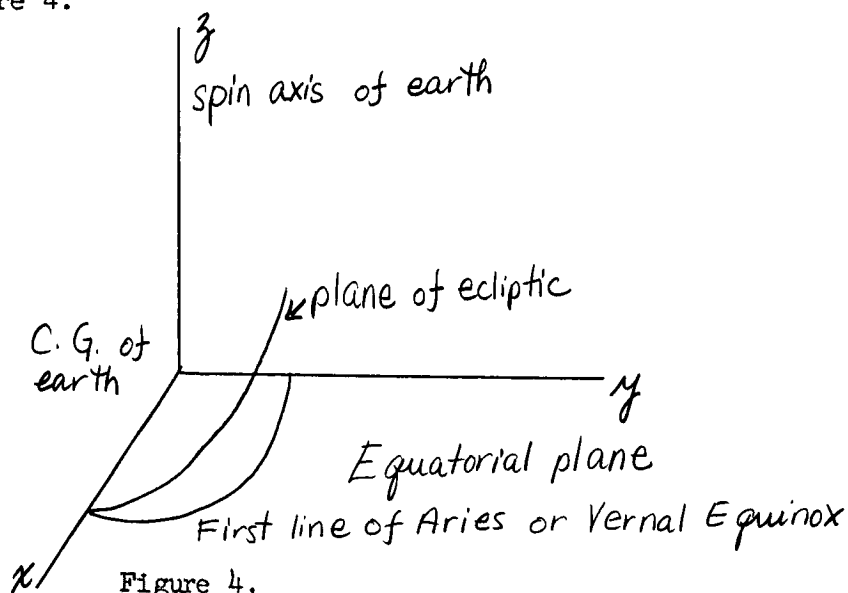
The fundamental errors we will be looking for are:

- (a) altitude error $E_{r_R} = \delta r_R$
- (b) North - South error $E_{\phi_R} = r_R \delta \phi_R$
- (c) East - West error $E_{\lambda_R} = r_R \cos \phi_R \delta \lambda_R$

where δr_R , $\delta \phi_R$, and $\delta \lambda_R$ are errors in range, latitude and longitude. Note for small flattening that $\phi_R - \phi_G = O(f)$.

Relative Geometry Between Satellite and Station

To describe the motion of a satellite we will use an inertial system as shown in Figure 4.



When we use the present instantaneous plane of the ecliptic such a coordinate system will be called the True Equatorial System of Date. When we use the Vernal Equinox of 1950.0 or 1963.5, for example, such a system will be called a Mean Equatorial System.

Clearly in such a system one has

$$\lambda_R(t) = \lambda_G + \lambda_{G_{T_0}}(t_0) + W_E(t - t_0)$$

where

$\lambda_R(t)$ = present longitude of tracking station

λ_G = longitude of tracking station with respect to Greenwich meridian

$\lambda_{G_{T_0}}(t_0)$ = longitude of Greenwich at time t_0 (this value can be found in the American Ephemeris)

W_E = Earth's rotational rate

Some Useful Coordinate Systems

Figure 5. shows a coordinate system X, Y, Z that has X, Y in the instantaneous plane of the orbit of a satellite and the Z axis perpendicular to this plane.

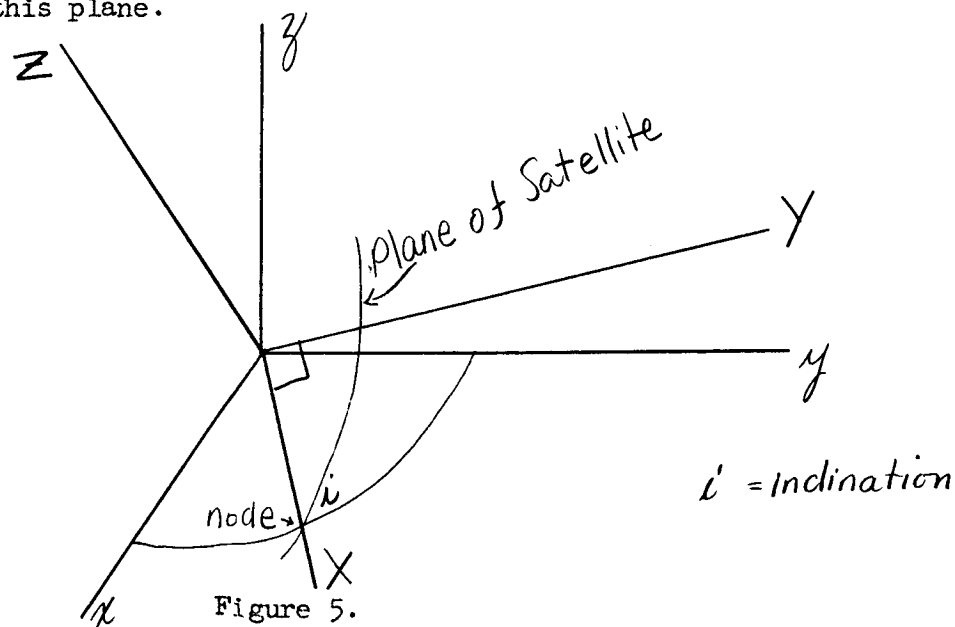


Figure 6. shows three positions of the tracking station in the X, Y, Z coordinate system. Note in this system the satellite moves in the X - Y plane. t_R means time of rise of the satellite, t_s is time of set, and t_c is time of closest approach of the satellite to the tracking station. $\vec{r}_R(t)$ is the position of the tracking station, while $\vec{\rho}(t)$ is the vector from the tracking station to the satellite. The unit vector H is in the direction of the satellite at time of closest approach. (The present H has nothing to do with geodial height.) $\beta(t)$ is the satellite argument of the latitude.

Figure 7. shows a cross section in the H - Z plane.

Figure 8. is a detailed picture of the X - Y plane. L is a unit vector in the X - Y plane perpendicular to H. Note that since H points to the direction of closest approach, we may assume that L is in the direction of motion of the satellite at time t_c . $\vec{r}_s(t)$ is the satellite position at time t.

Geometry of Path

Defining θ and e as in Figure 7., we now wish to prove the useful formula:

$$\sin \theta = (1 - r_{R,s}^2) / \left[\sqrt{1 - r_{R,s}^2 \cos^2 e} + r_{R,s} \sin e \right]$$

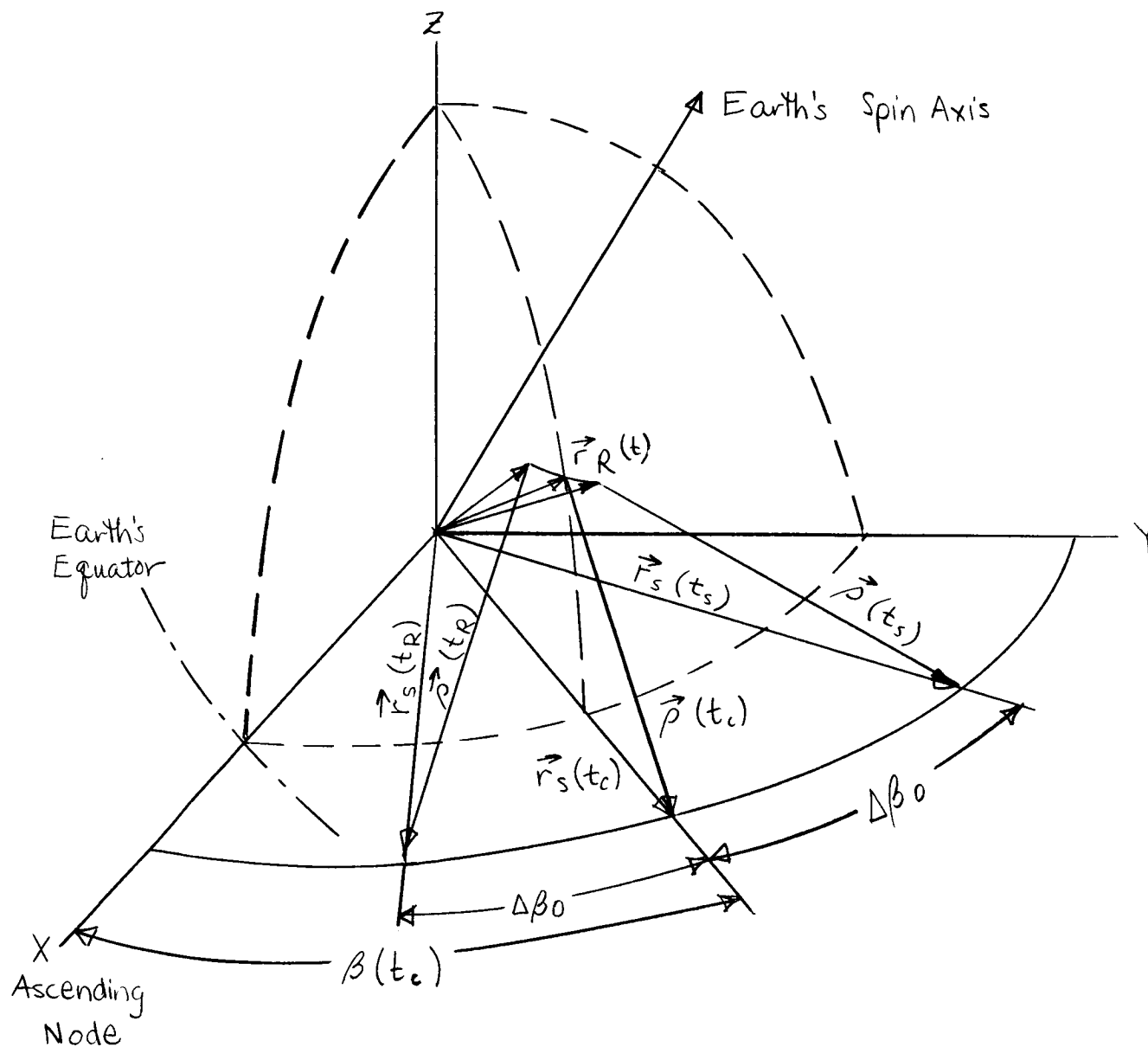
with

$$r_{R,s} = |\vec{r}_R(t_c)| / \rho_c ; \quad \rho_c = |\vec{\rho}(t_c)|$$

Proof:

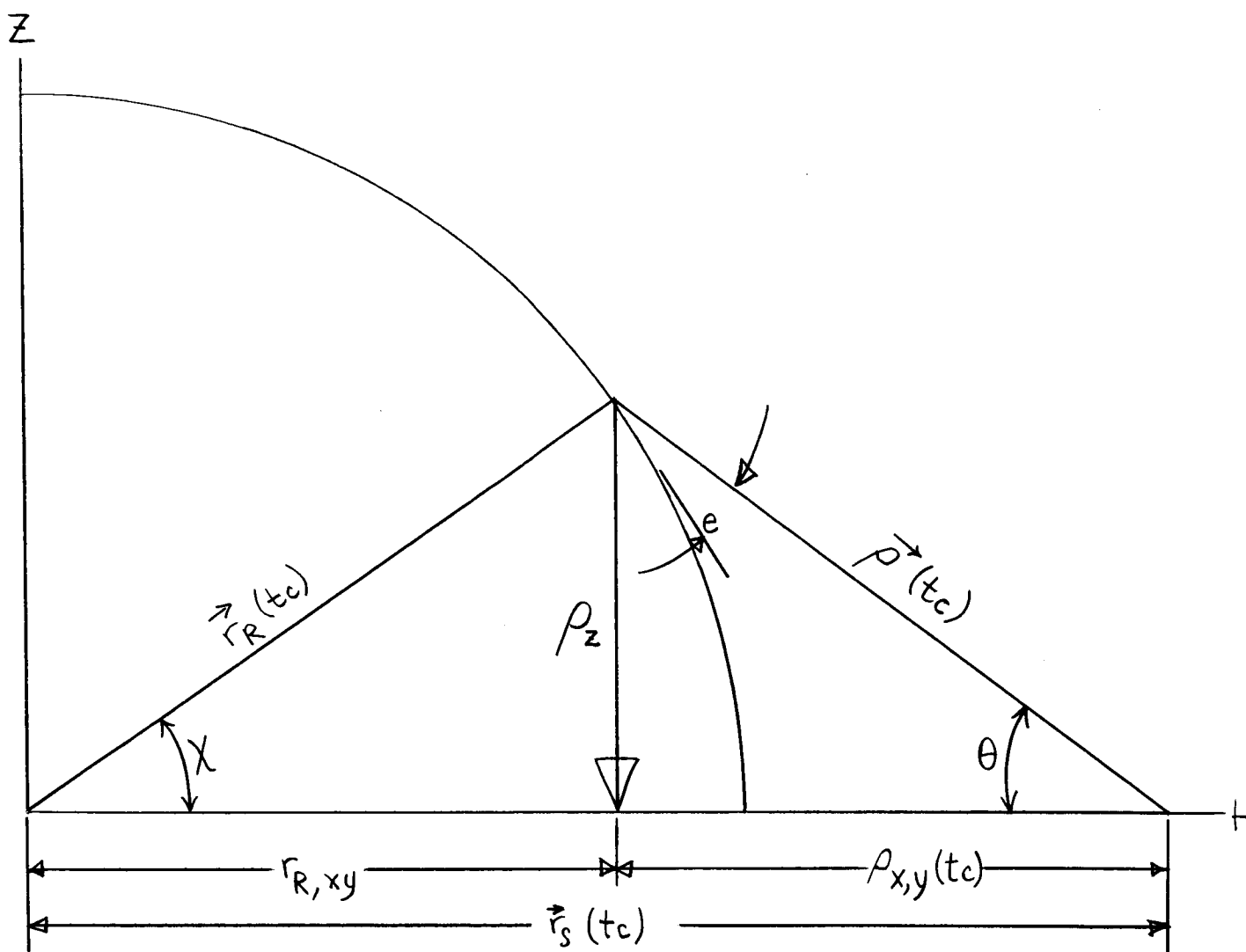
From Figure 8., setting $r_R = |\vec{r}_R(t_c)|$, $r_s = |\vec{r}_s(t_c)|$ one has:

$$r_R^2 = r_s^2 + \rho_c^2 - 2r_R\rho_c \cos \theta$$



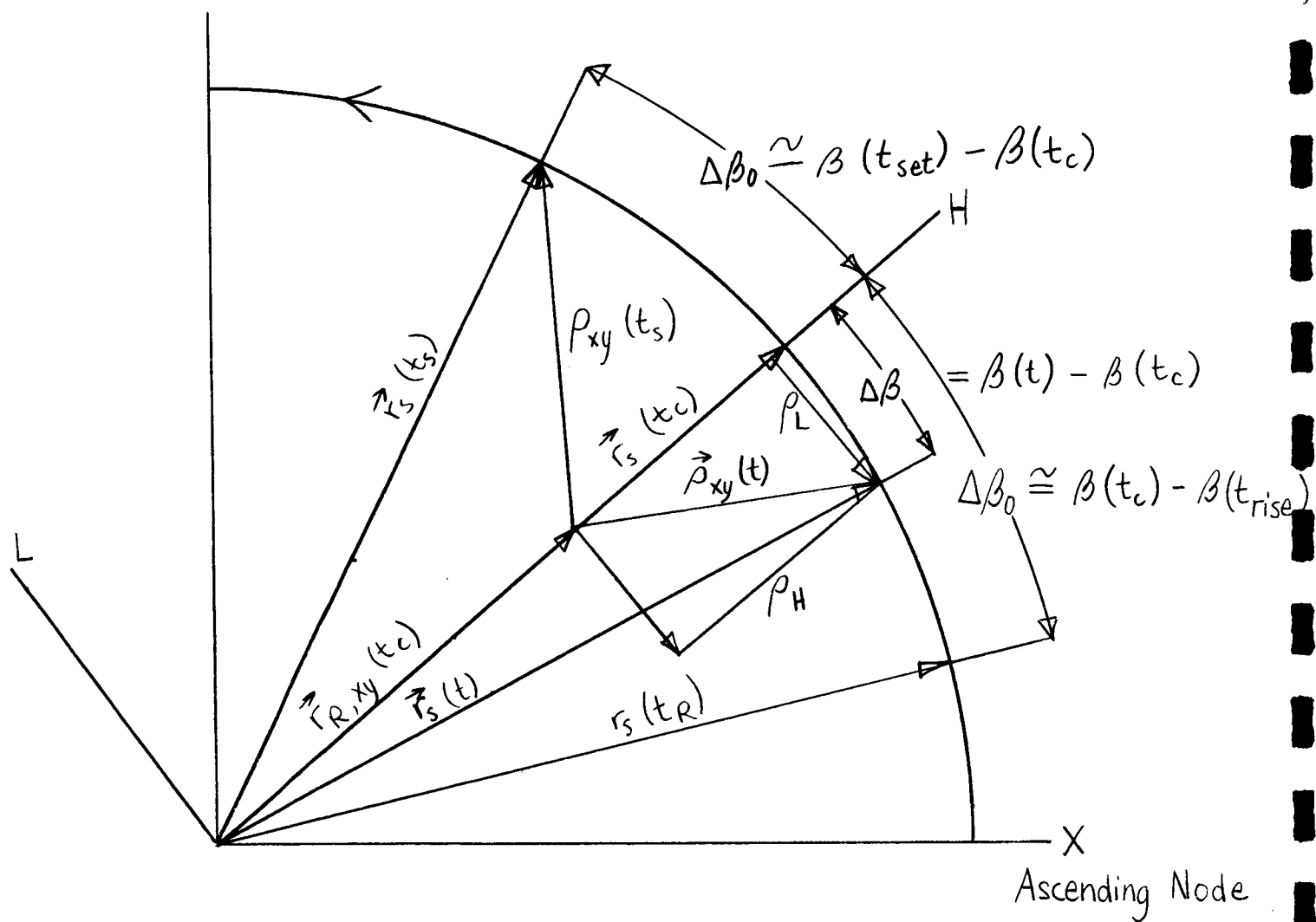
Geometry During Satellite Pass ($x - y$ plane = Orbital Plane)

Figure 6.



Geometry at Time of Minimum Slant Range
 (H - Z Plane, Satellite motion into page)

Figure 7.



Geometry of Pass (Orbital Plane)

Figure 8.

and

$$\frac{\sin \theta}{r_R} = \frac{\sin (\pi/2 + e)}{r_s} = \frac{\cos e}{r_s}$$

Now defining

$$\rho_s = \rho_c/r_s \quad \text{and} \quad r_{R,s} = r_R/r_s$$

one has

$$(A) \quad r_{R,s}^2 = 1 + \rho_s^2 - 2 \rho_s \cos \theta$$

$$(B) \quad \sin \theta = \rho_s \cos e$$

Solving (A) for ρ_s one obtains:

$$\rho_s = \cos \theta - \sqrt{r_{R,s}^2 - \sin^2 \theta}$$

Substituting in (B) one obtains the desired answer.

Pseudo Azimuth and Elevation

With the usual definitions of azimuth A_z and elevation E_ℓ , one encounters certain difficulties. Thus with an overhead satellite, azimuth changes by 180° . To prevent this difficulty two new quantities "pseudo azimuth" a_z and "pseudo elevation" e have been introduced.

They are shown in Figures 9. and 10., and in Table 1, where formulas for converting from azimuth to pseudo-azimuth and from elevation to pseudo-elevation are given.

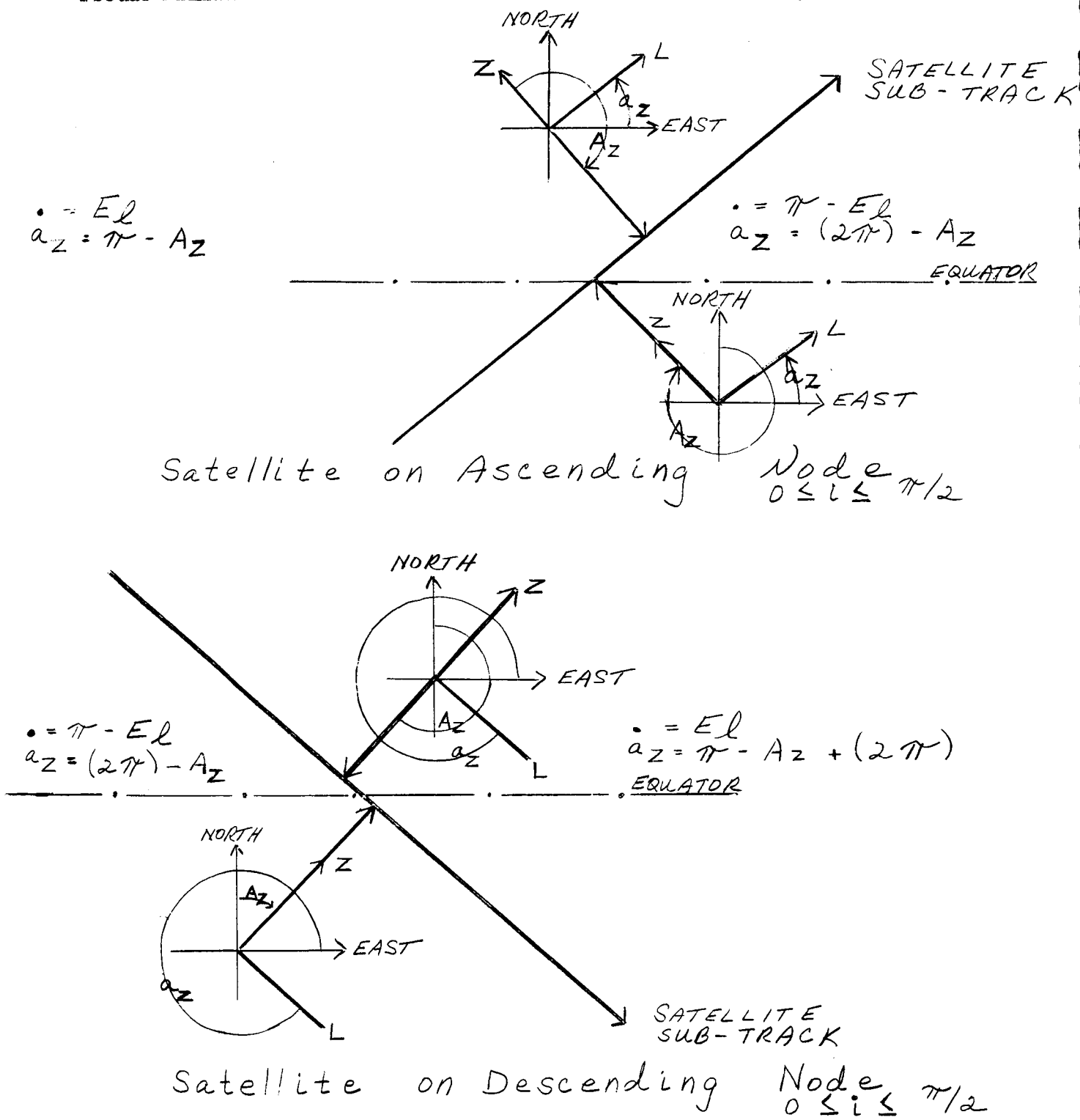
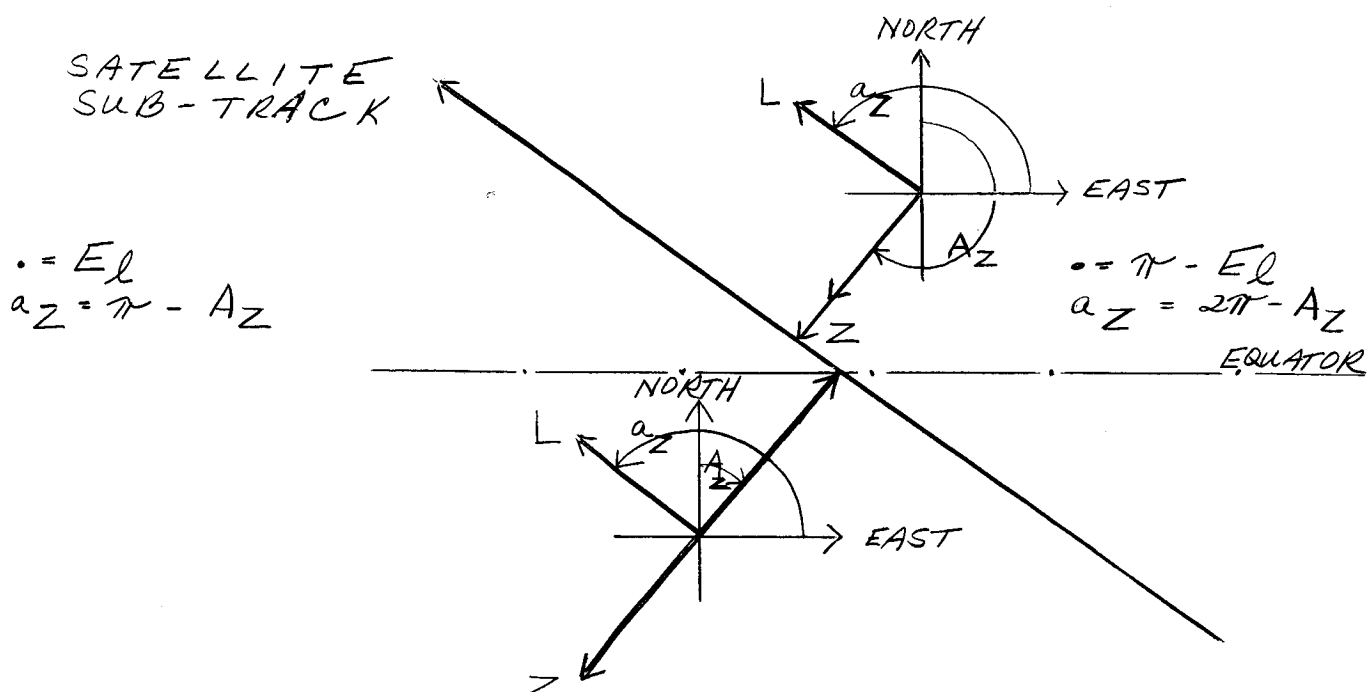
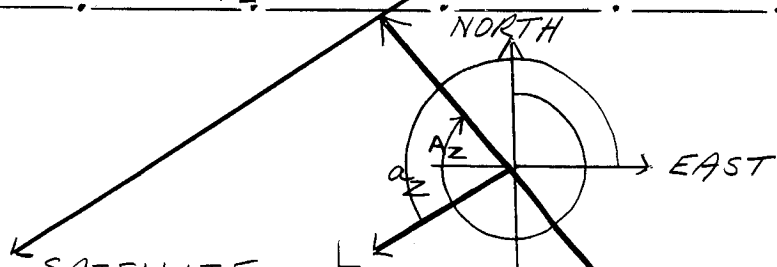
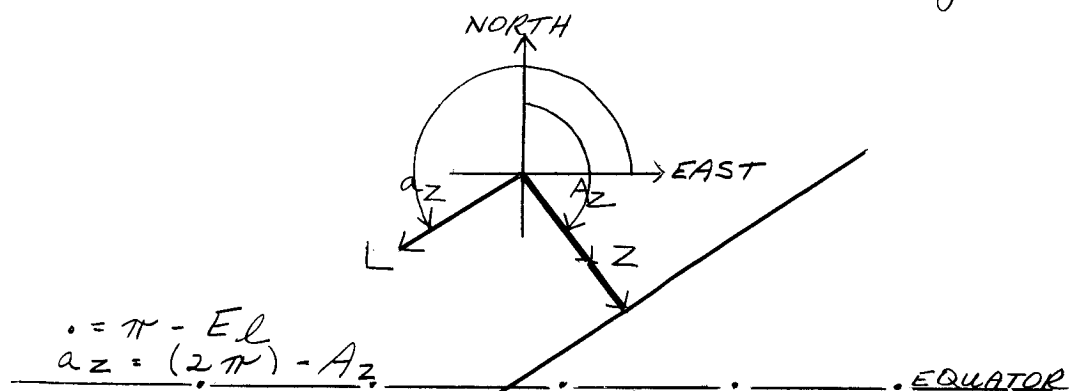


Figure 9.

Pseudo Elevation and Azimuth (Advance Satellite Motion)



Satellite on Ascending Node $\pi/2 < i \leq \pi$



SATELLITE SUB-TRACK
 Satellite on Descending Node

$\bullet = El$
 $a_z = \pi - A_z + (2\pi)$

Figure 10.

Pseudo Elevation and Azimuth (Retrograde Satellite Motion)

PSEUDO ELEVATION AND AZIMUTH

Azimuth	Satellite Inclination			
	$0 \leq i \leq \pi/2$		$\pi/2 < i \leq \pi$	
$0 \leq A_z < \pi/2$	$e = \pi - E_\ell$	$a_z = -A_z$	$e = E_\ell$	$a_z = \pi - A_z$
$\pi/2 \leq A_z < \pi$	$e = E_\ell$	$a_z = \pi - A_z$	$e = \pi - E_\ell$	$a_z = -A_z$
$\pi \leq A_z < \frac{3\pi}{2}$	$e = E_\ell$	$a_z = \pi - A_z$	$e = \pi - E_\ell$	$a_z = -A_z$
$\frac{3\pi}{2} \leq A_z < 2\pi$	$e = \pi - E_\ell$	$a_z = -A_z$	$e = E_\ell$	$a_z = \pi - A_z$

 E_ℓ = Elevation A_z = Azimuth e = Pseudo Elevation a_z = Pseudo Azimuth

Table 1.

We will now derive the coordinates of $\vec{\rho}(t)$ in the H - L - Z coordinate system.

Setting $\vec{\rho}(t) = (\rho_Z, \rho_H, \rho_L)$ we first have, (see Figure 7.), since ρ_Z does not change with time,

$$\rho_Z = -\rho_c \sin \theta = -r_s(t_c) \rho_s \sin \theta$$

where we have defined $\rho_s \equiv \rho_c / r_s(t_c)$.

For ρ_H , one first has from Figure 8.

$$\rho_H = |r_s(t)| \cos \Delta \beta - r_{R,xy}(t_c).$$

Since we are dealing with nearly circular motion, we have:

$$(A) \quad |r_s(t)| = r_s(t_c) + o(\epsilon) \quad r_s(t_c) = |\vec{r}_s(t_c)|$$

and from Figure 7., we have:

$$(B) \quad r_{R,xy}(t_c) = r_s(t_c) - \rho_c \cos \theta = r_s(t_c) [1 - \rho_s \cos \theta]$$

Substituting (A) and (B) into the expression for ρ_H , one has:

$$\rho_H = r_s(t_c) [\cos \Delta \beta(t) - (1 - \rho_s \cos \theta)] + \text{1st order}$$

Similarly for ρ_L , one obtains with the use of (A)

$$\rho_L = |r_s(t)| \sin \Delta \beta(t) = r_s(t_c) \sin \Delta \beta(t) + \text{1st order}$$

Using the notation

$$\alpha = 1 - \rho_s \cos \theta$$

$$c(t) = 1 - \cos \Delta \beta(t)$$

one can write the above formulas as:

$$\vec{\rho}(t) = \begin{pmatrix} \rho_H \\ \rho_L \\ \rho_Z \end{pmatrix} = r_s(t_c) \begin{pmatrix} 1 - \alpha - c(t) \\ \sin \Delta \beta(t) \\ - \rho_s \sin \theta \end{pmatrix}$$

correct through the zeroth order.

For the scalar slant range, one obtains:

$$\rho(t) = \sqrt{\vec{\rho}(t) \cdot \vec{\rho}(t)} = r_s(t_c) \sqrt{\rho_s^2 + 2\alpha c(t)}.$$

(Please note that in the present series of notes that sometimes the tracking station will be given coordinates X_T, Y_T, Z_T , standing for tracking, and sometimes the notation X_R, Y_R, Z_R will be used, standing for receiving. Both subscripts T and R refer to the same thing.)

We have previously defined the tracking errors E_{r_T}, E_{ϕ_T} and E_{λ_T} . (These are about 1/4 kilometer in magnitude.) The corresponding formulas in H-L-Z coordinates are:

$$E_{H_T} = r_{R,s} E_{r_T} + \rho_s \left[\sin e E_{r_T} - \cos e \cos a_z E_{\phi_T} + \cos e \sin a_z E_{\lambda_T} \right]$$

$$E_{L_T} = \sin a_z E_{\phi_T} + E_{\lambda_T} \cos a_z$$

$$E_{Z_T} = r_{R,s} \left[\cos a_z E_{\phi_T} - \sin a_z E_{\lambda_T} \right] + \rho_s \left[\sin e \cos a_z E_{\phi_T} - \sin e \sin a_z E_{\lambda_T} + \cos e E_{r_T} \right]$$

These formulas are obtained by a rotation. It is assumed that during the time of pass that the station does not change its position.

Errors in Satellite Position

We now discuss the motion of a satellite. We begin by giving a few standard definitions.

a = semi-major axis (scaled by R_0)

e = eccentricity

i = inclination

ω = argument of perigee

Ω = argument of node

M = mean anomaly

M_0 = mean anomaly at epoch

$\dot{M} = n_0$ = mean motion

f = true anomaly

$\beta = f + \omega$ = argument of latitude.

Some standard formulas from two-body motion that will constantly be used are:

$$r_s(t) = a(1 - \epsilon^2)/(1 + \epsilon \cos(\beta - \omega))$$

$$\tan f = \frac{\sqrt{1 - \epsilon^2} \sin E}{(\cos E - \epsilon)}$$

$$\sin \phi = \sin i \sin \beta$$

$$\tan(\lambda - \Omega) = \cos i \tan \beta.$$

We will assume that we already have a good idea of the coefficients J_n , C_n^m , S_n^m in the earth's potential, and that we have a good method for integrating the equations of motion. Thus we will know the orbital parameters a , e , i , etc. quite accurately.

The problem of principal interest will be to determine small changes δ_a , δ_e , δ_i , etc. in the orbital parameters caused by changes ΔJ_n , ΔC_n^m , ΔS_n^m in the earth's potential.

Thus if we define

$$\Delta U = \frac{K}{R_0} \frac{1}{r} \left[\sum_n \Delta J_n \frac{P_n(\sin \phi)}{r^n} + \sum_n \sum_m \frac{P_n^m(\sin \phi)}{r^n} [C_n^m \cos m \lambda + S_n^m \sin m \lambda] \right].$$

The equations of motion for the small changes in the orbital parameters

(assuming small eccentricity) become:

$$\begin{aligned}\delta \dot{a} &= \frac{2}{n_0 a} \frac{\partial \Delta U}{\partial \beta} + o(\epsilon) \\ \delta \dot{e} &= \frac{1}{n_0 a^2} \left[\sin(\beta - \omega) \frac{\partial \Delta U}{\partial a} + \frac{2}{a} \cos(\beta - \omega) \frac{\partial \Delta U}{\partial \beta} \right] + o(\epsilon) \\ \frac{d}{dt} \delta i &= \frac{1}{n_0 a^2} \cot \beta \frac{\partial \Delta U}{\partial i} + o(\epsilon) \\ \sin r_0 \delta \dot{\Omega} &= \frac{1}{n_0 a^2} \frac{\partial \Delta U}{\partial i} + o(\epsilon) \\ \epsilon \delta \dot{\omega} &= \frac{1}{n_0 a} \left[-\cos(\beta - \omega) \frac{\partial \Delta U}{\partial a} + \frac{2}{a} \sin(\beta - \omega) \frac{\partial \Delta U}{\partial \beta} \right] + o(\epsilon) \\ \frac{d}{dt} (\delta M + \delta \omega) &= -\frac{3}{2} \frac{\delta a}{a} n_0 - \frac{2}{n_0 a} \frac{\partial \Delta U}{\partial a} - \cos i \delta \dot{\Omega} = o(\epsilon)\end{aligned}$$

(This last combination of variables is to avoid terms containing $1/\epsilon$.)

Note that in these equations if $\Delta U = 0$, then a solution would be:

$$\begin{aligned}\delta a &= \delta a_0 & \epsilon \delta \omega &= \epsilon \delta \omega_0 \\ \delta e &= \delta e_0 & \delta \Omega &= \delta \Omega_0 \\ \delta i &= \delta i_0 & \delta M + \delta \omega &= \delta M_0 - \frac{3}{2} \frac{\delta a_0}{a_0} n_0 (t - t_0) + \\ & & & \text{terms of higher order}\end{aligned}$$

where the subscript 0 means initial (constant) values. Thus in order to fit data better, it is always possible to add constants δa_0 , δe_0 , δi_0 , etc.

As an example of the use of the above equation, we note that if a change ΔJ_3 is made, the orbital parameters change in the following way:

$$\begin{aligned}\delta_a &= \text{second order change, i.e., no material change} \\ \delta \epsilon &= \frac{1}{2} \frac{\Delta J_3}{J_2} \frac{\sin i}{a} \sin \omega + o\left(\epsilon \frac{\Delta J_3}{J_2}\right) \\ \epsilon \delta \omega &= \frac{1}{2} \frac{\Delta J_3}{J_2} \frac{\sin i}{a} \cos \omega + o\left(\epsilon^2 \frac{\Delta J_3}{J_2}\right) \\ \delta i, \delta \Omega, \delta (M + \omega) &\text{ all } o\left(\epsilon \frac{\Delta J_3}{J_2}\right)\end{aligned}$$

General Rules to Order Unity

One can list the following effects of changes ΔJ_n , Δc_n^m , Δs_n^m on the orbital parameters:

1. ΔJ_n for n odd give long period changes in ϵ and ω .
2. ΔJ_n for n even give
 - a) secular changes in $\omega, \Omega, M + \omega$
 - b) long period changes in $\omega, M + \omega$
 - c) short period changes in all orbital parameters.
3. $\Delta c_n^m, \Delta s_n^m$ give changes, the period being

$$P = \frac{\text{sidereal day} - \text{nodal rate}}{m}$$

Transforming Changes into H - L - Z Coordinate System

As a first step to transferring changes into the H-L-Z coordinate

system, we will transform into the moving coordinate system δr , δl , Z (pointing to the satellite) shown in Figure 11. δl is usually called long track, and δr cross track. They are in the plane of the orbit, while Z is perpendicular to the plane of the orbit.

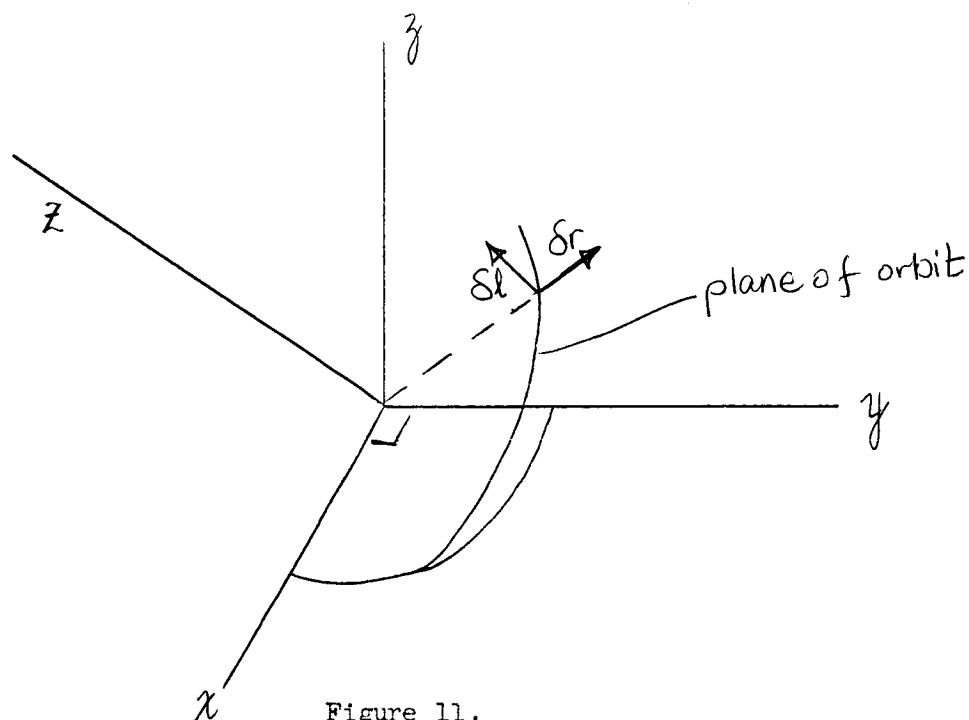


Figure 11.

The basic formulas are:

$$\begin{aligned}
 \text{(A)} \quad \delta r_s &= \delta a - a \left[\delta \epsilon \cos(\beta - \omega) + \epsilon \delta \omega \sin(\beta - \omega) \right] + o(\epsilon) \\
 \text{(B)} \quad \delta l &= a \left[(\delta M + \delta \omega) + 2 \delta \epsilon \sin(\beta - \omega) - \right. \\
 &\quad \left. 2 \epsilon \delta \omega \cos(\beta - \omega) + \delta \Omega \cos i \right] + o(t) \\
 \text{(C)} \quad \delta Z &= a \left[\delta i \sin \beta - \delta \Omega \cos i \cos \beta \right]
 \end{aligned}$$

To derive (A), one proceeds as follows:

$$\delta r_s = \delta \left[\frac{a(1 - \epsilon^2)}{1 + \epsilon \cos(\beta - \omega)} \right] = \frac{\delta a}{1 + \epsilon \cos(\beta - \omega)} - \frac{a(1 - \epsilon^2)}{[1 + \epsilon \cos(\beta - \omega)]^2} \times Q$$

with

$$Q = \delta \epsilon \cos(\beta - \omega) - \epsilon \sin(\beta - \omega) (\delta \beta - \delta \omega).$$

Now since

$$\delta \beta = \delta(M + \omega) + 2 \left[\delta \epsilon \sin(\beta - \omega) - \epsilon \delta \omega \cos(\beta - \omega) \right] + o(\epsilon); \quad \epsilon \delta \beta \text{ can be neglected.}$$

Thus to $O(\epsilon)$ formula (A) is verified.

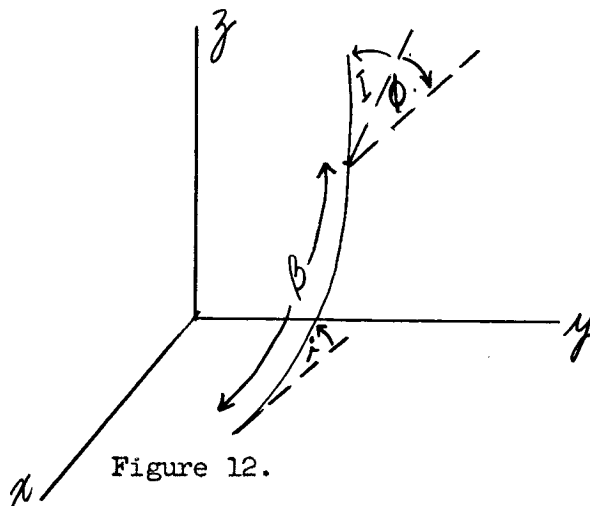


Figure 12.

To derive formulas (B) and (C), one must first use the local inclination i of the orbit, and the local precession of the node $r_s \cos \phi \delta \Omega$ at (ϕ, λ) to obtain:

$$\begin{aligned}\delta \ell &= r_s \delta \Omega \cos \phi \cos I + f_\ell(\delta \beta) \\ \delta z &= -r_s \delta \Omega \cos \phi \sin I + f_z(\delta \beta)\end{aligned}$$

Now using the formulas:

$$\begin{aligned}\tan I &= \tan i \cos \beta \\ \cos i &= \cos I \cos \phi\end{aligned}$$

and looking at Figures 6., 7., and 8. to obtain $f_\ell(\delta \beta)$, $f_z(\delta \beta)$, one finds

$$\begin{aligned}\delta \ell &= r_s \delta \Omega \cos i + r_s \delta \beta \\ \delta z &= -r_s \delta \Omega \sin i \cos \beta + r_s \sin \beta \delta i\end{aligned}$$

Finally using the previously cited formula for $\delta \beta$, one obtains the desired result.

We begin by listing the deviations that would take place in the $\delta r_s, \delta \ell_s, \delta z_z$ coordinate system if we have a deviation ΔJ_3 in J_3 .

First setting

$$\begin{aligned}\delta A_c &= -a \left[\delta \epsilon_0 \cos \omega(t_c) - (\epsilon_0 \delta \omega_0) \sin \omega(t_c) \right] \\ \delta B_c &= -(\Delta J_3 / 2J_2) \sin i - \\ &\quad a \left[\delta \epsilon_0 \sin \omega(t_c) + (\epsilon_0 \delta \omega_0) \cos \omega(t_c) \right],\end{aligned}$$

we find the deviations at time t_c as:

$$\begin{aligned}\delta r_c &= \delta a_0 + \delta A_c \cos \beta_c + \delta B_c \sin \beta_c \\ \delta \ell_c &= a \left[\delta(M_0 + \omega_0) + \delta \Omega_0 \cos i \right] - 2/3 \delta B_c - \\ &\quad 2 \delta A_c \sin \beta_c + 2 \delta B_c \cos \beta_c \\ \delta z_c &= -\delta \Omega_0 \sin i \cos \beta_c + \delta i_0 \sin \beta_c,\end{aligned}$$

and finally the deviation at an arbitrary time (assumed during one fixed pass) is: (with $c(\Delta \beta) = 1 - \cos(\Delta \beta)$)

$$\begin{aligned}\delta r_s &= \delta r_c + \left[-\delta A_c \sin \beta_c + \delta B_c \cos \beta_c \right] \sin \Delta \beta - \\ &\quad \left[\delta A_c \cos \beta_c + \delta B_c \sin \beta_c \right] c(\Delta \beta) \\ \delta \ell_s &= \delta \ell_c - 2 \left[\delta A_c \cos \beta_c + \delta B_c \sin \beta_c \right] \sin \Delta \beta - \\ &\quad 2 \left[-\delta A_c \sin \beta_c + \delta B_c \cos \beta_c \right] c(\Delta \beta) \\ \delta z_s &= \delta z_c + \left[\delta \Omega_0 \sin i \sin \beta_c + \delta r_0 \cos \beta_c \right] \sin \Delta \beta - \\ &\quad \delta z_c c(\Delta \beta)\end{aligned}$$

The things to be stressed in these equations are:

1. Always have changes in elements.
2. Change in character of time dependence.
3. When the elements have only long term effects, then the examples are general.

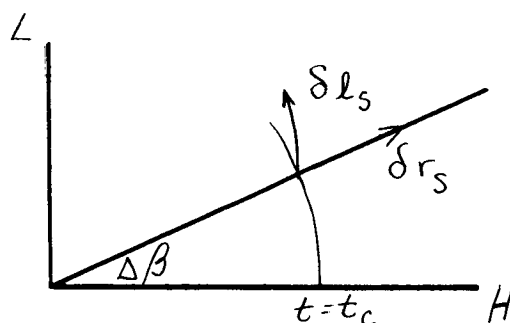


Figure 13.

It is clear from Figure 13., that the δr_s , δl_s , Z coordinate system agrees with the H-L-Z coordinate system at $t = t_c$. For other times, one must rotate through an angle of $\Delta\beta$ to put everything in H-L-Z coordinates. Using our definition for $C(\Delta\beta)$, one obtains for the deviations:

$$H_s(t) = \delta r_s - \delta l_s \sin \Delta\beta - \delta r_s C(\Delta\beta)$$

$$L_s(t) = \delta l_s + \delta r_s \sin \Delta\beta - \delta l_s C(\Delta\beta)$$

$$Z_s(t) = \delta Z_s(t).$$

Changes in Slant Range Vector

We can now combine our previous results into the change $\delta \vec{\rho}$ in the slant range vector, where $\delta \vec{\rho} = \delta \vec{r}_s - \delta \vec{r}_R$

One finds for small $\Delta \beta$ that in the H-L-Z coordinate system one has:

$$\delta \vec{p}(\Delta \beta) = \delta \vec{p}_c + \delta \vec{p}_1 \sin(\Delta \beta) + \delta \vec{p}_2 c(\Delta \beta)$$

with the H-L-Z components of $\delta \vec{p}_c$, $\delta \vec{p}_1$ and $\delta \vec{p}_2$ being

$$\delta \vec{p}_c = \begin{pmatrix} \delta r_c - E_{H_T} \\ \delta \ell_c - E_{L_T} \\ \delta z_c - E_{Z_T} \end{pmatrix}$$

$$\delta \vec{p}_2 = \begin{pmatrix} -\delta r_c + 3\delta A_c \cos \beta_c + 3\delta B_c \sin \beta_c \\ -\delta \ell_c \\ -\delta z_c \end{pmatrix}$$

$$\delta \vec{p}_1 = \begin{pmatrix} -\delta \ell_c - \delta A_c \sin \beta_c + \delta B_c \sin \beta_c \\ \delta r_c - 2(\delta A_c \cos \beta_c + \delta B_c \sin \beta_c) \\ \delta \Omega_0 \sin i \sin \beta_c + \delta r_0 \cos \beta_c \end{pmatrix}$$

Types of Data

1. Radar that measures both range and angle accurately. If such a system existed it would give one all nine components of $\delta \vec{p}_c$, $\delta \vec{p}_1$, $\delta \vec{p}_2$.

2. Range only from radar.
3. Angle only from optical instruments.
4. Range rate or Doppler. For a study of this last type of data see W.H. Guier, Studies on Doppler Residuals - 1: Dependence on Satellite Orbit Error and Station Position Error. TG - 503, June 1963, Applied Physics Laboratory, The Johns Hopkins University, Silver Springs, Maryland.

Range Data

We will begin by studying range data. Note that

$$\rho \delta \rho = (1/2) \delta (\vec{\rho} \cdot \vec{\rho}) = \vec{\rho} \cdot \delta \vec{\rho}$$

We previously derived formulas for $\vec{\rho}$ and ρ in the H-L-Z coordinates, so that one easily obtains the following formula for the range only data:

$$\begin{aligned} \frac{\rho \delta \rho}{r_s} = & \rho_s [\cos \theta \delta \rho_{c_H} - \sin \theta \delta \rho_{c_Z}] \\ & + [\delta \rho_{c_L} + \rho_s (\cos \theta \delta \rho_{L_H} - \sin \theta \delta \rho_{L_Z})] \sin \Delta \beta \\ & + [2 \delta \rho_{L_L} - \delta \rho_{c_H} + \rho_s (\cos \theta \delta \rho_{2_H} - \sin \theta \delta \rho_{2_Z})] \cos \Delta \beta \\ & + o((\Delta \beta)^2) \end{aligned}$$

Note in these formulas $\delta \rho_{c_H}$ means the H component of $\delta \vec{\rho}_c$,

$\delta \rho_{L_L}$ means the L component of $\delta \vec{\rho}_L$, etc.

Figure 14. shows the behavior of a few functions on $\Delta \beta$. With this type of analysis, one can separate out the various components in $\rho \delta \rho / r_s$.

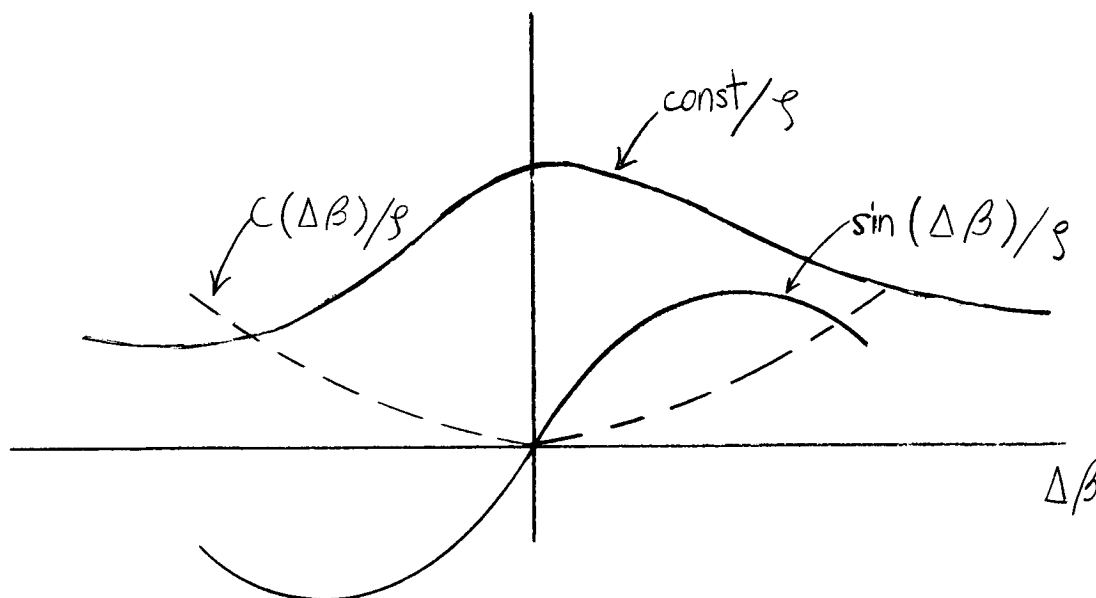


Figure 14.

Table Two

Table 2 lists the three principal types of data, namely, range only, Doppler, and angle only data versus the three types of time dependence, constant, anti-symmetric and symmetric. The equations for range only were previously derived. Those for Doppler and for angle only are merely listed. To derive the latter two, one should use the formulas: $\hat{\rho} = \vec{\rho}/\rho$, the unit slant range vector]

$$\delta(\hat{\rho}) = \delta\left(\frac{\vec{\rho}}{\rho}\right) = \frac{\delta\vec{\rho}}{\rho} - \frac{\vec{\rho}\delta\rho}{\rho^2} = \frac{1}{\rho^3} \left[\rho^2 \delta\vec{\rho} - \vec{\rho}(\rho\delta\rho) \right]$$

$$\frac{d}{dt}(\delta\rho) = \frac{d}{dt}\left(\frac{1}{\sqrt{\rho^2}} \rho \delta\rho\right) = \frac{1}{\rho^3} \left[-\frac{1}{2} \rho \delta\rho \frac{d}{dt} \rho^2 + \rho^2 \frac{d}{dt} \rho \delta\rho \right]$$

It should be remarked that although $\delta \hat{\rho}$ has really only two components (those perpendicular to $\hat{\rho}$); that in Table 2 these two components have been resolved along the three coordinate axes H-L-Z.

Several interesting facts are apparent from the table.

1. Any competent system will resolve at least through the anti-symmetric data.
2. Doppler and range only data are about equivalent in data content.
3. Optical data and radio interferometer data as a function of time will yield a vast amount of information.

Collections of Passes

So far we have only been concerned about one pass. When one considers several passes statistical questions naturally arise. Since one is concerned with very many parameters, it is essential to have a large amount of data to avoid singular variance co-variance matrices. It is felt that in the present state of affairs that the signal to noise ratio is high, so that elaborate statistical routines are not necessary. In fact, besides least squares some sort of minimax routine would be useful.

Typical Problems

Some typical problems that the foregoing theory could be, and is being applied to are:

Table Two

32.

Type of Data Time Dependence	$\left(\frac{P(t)}{r_a}\right)^3 \delta P(t)$	$-\frac{1}{n_0} \left(\frac{P(t)}{r_a}\right)^3 \frac{d}{dt} \delta P$	$\frac{P(t)}{r_a} \delta \hat{P} = \begin{pmatrix} H \\ L \\ C \end{pmatrix}$
(constant)	$P_a^3 [\cos \theta \delta P_{H_C} - \sin \theta \delta P_{Z_C}]$	$P_a^2 \delta P_{L_C} + O(P_a^3)$	$P_a^2 \begin{pmatrix} \sin \theta [\sin \theta \delta P_{H_C} + \cos \theta \delta P_{Z_C}] \\ \delta P_{L_C} \\ \sin \theta [\cos \theta \delta P_{H_C} + \sin \theta \delta P_{Z_C}] \end{pmatrix} + O(P_a^3)$
$\sin \Delta \beta$	$P_a^2 \delta P_{L_C} + O(P_a^3)$	$-P_a [\cos \theta \delta P_{H_C} - \sin \theta \delta P_{Z_C}]$	$P_a \begin{pmatrix} -\cos \delta P_{L_C} \\ -[\cos \theta \delta P_{H_C} - \sin \theta \delta P_{Z_C}] \\ \sin \theta \delta P_{L_C} \end{pmatrix} + O(P_a^2)$
$c(\Delta \beta) = \frac{1 - \cos(\Delta \beta)}{2}$	$2P_a [\cos \theta \delta P_{H_C} - \sin \theta \delta P_{Z_C}]$	$O(P_a^2)$	$\begin{pmatrix} 2\delta P_{H_C} \\ 0 \\ 2\delta P_{Z_C} \end{pmatrix} + O(P_a)$

1. Location of islands in a large body of water, e.g., Hawaii.

Since one can track satellites both from North America and Hawaii, it would be possible to locate the tracking stations in Hawaii very accurately.

2. To determine coefficients J_n , S_n^m , C_n^m in the earth's potential.
3. To locate, for example, the European datum with respect to the C. G. of the earth and with respect to the North America datum.

Other Problems

A list of references is attached of other problems such as refraction problems in radar data, how to eliminate spurious data points, etc.

References

34.

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PRACTICAL ASTRONOMY

by

Professor P. Herget

I. Celestial Sphere and Spherical Coordinates.

Astronomy is the science which deals with the bodies of the universe and in particular with their positions, motions, constitution and evolution. The observation of those bodies such as stars (self-luminous bodies), planets (bodies revolving around a star), and satellites (bodies revolving around a planet) is the concern of practical astronomy.

All the celestial bodies will appear to an observer on the surface of the Earth as lying on a spherical shell overhead, rotating about an axis through the observer. This imaginary sphere of indefinite radius is termed the celestial sphere and all the bodies in the universe are assumed to be on the surface of the sphere. Thus it is natural to use the spherical coordinates system to define the positions of stars, planets or satellites in terms of the two angles of the system.

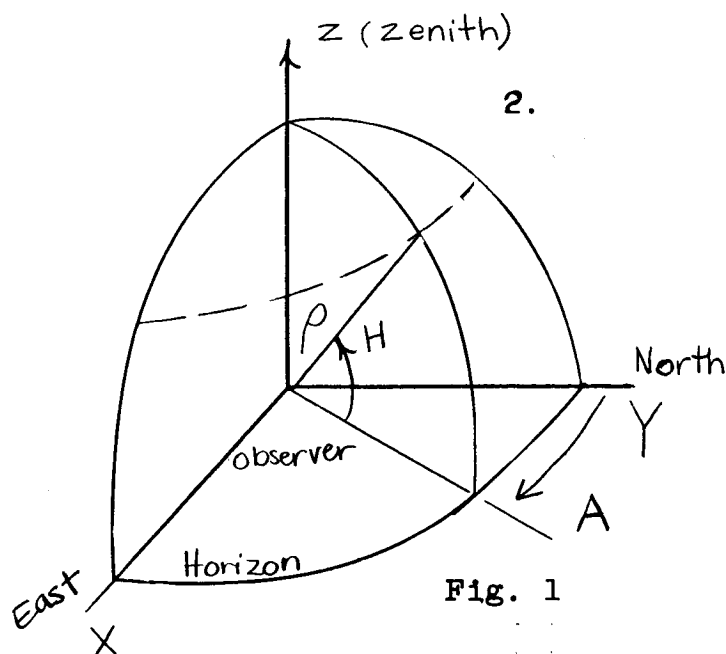
One system used for tracking artificial satellites, which we shall call the local coordinates system, has the zenith as the local vertical and the horizon (basin of mercury) as the base plane. Choose any point on the horizon as the zero point -- I prefer the north point; we then have spherical coordinates ρ , A , H (Fig. 1).

Draw a great circle which is a circle on and has the same radius as the celestial sphere through the satellite. Its position is then located by two angles. The angle H measured from the horizon is called altitude; the angle A on the horizon measured from the north point toward the

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east is called azimuth.
The cartesian coordinates
X Y Z are given by

$$\begin{aligned} X &= \rho \cos H \sin A \\ Y &= \rho \cos H \cos A \\ Z &= \rho \sin H \end{aligned} \quad (1)$$



For astronomical observation, radius ρ is unobserved.

This system has been used in Project Vanguard, known as MiniTrack system, and in Project Mercury. In the former, two direction cosines l and m , oriented along N - S and E - W respectively, are measured. Thus

$$\begin{aligned} l &= \cos H \sin A \\ m &= \cos H \cos A \end{aligned} \quad (2)$$

The third direction cosine is calculated from l and m

$$n = \sin H = \sqrt{1 - l^2 - m^2},$$

which becomes highly indeterminate when the subject is near the horizon.

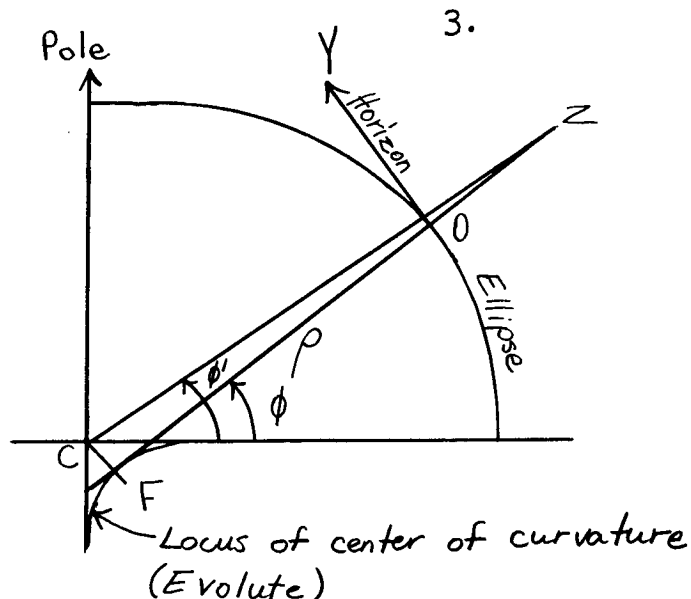
Since the Earth's surface is approximately elliptical, the local vertical should be oriented along the radius of curvature as shown in the figure. The center of the Earth as measured from the local coordinates X Y Z on the surface

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is positioned at

$$\begin{aligned} X &= 0 \\ Y &= \rho \sin(\phi - \phi') \\ Z &= -\rho \cos(\phi - \phi') \end{aligned} \quad (3)$$

The difference between the two angles is actually very small because the major axis of the Earth ellipse is only about 13 miles longer than its minor axis.



C - center of earth
F - center of curvature

Fig. 2

The angle ϕ is called the astronomical latitude and ϕ' the geocentric latitude. The former is preferred in astronomical observation, and the latter is used in map-making.

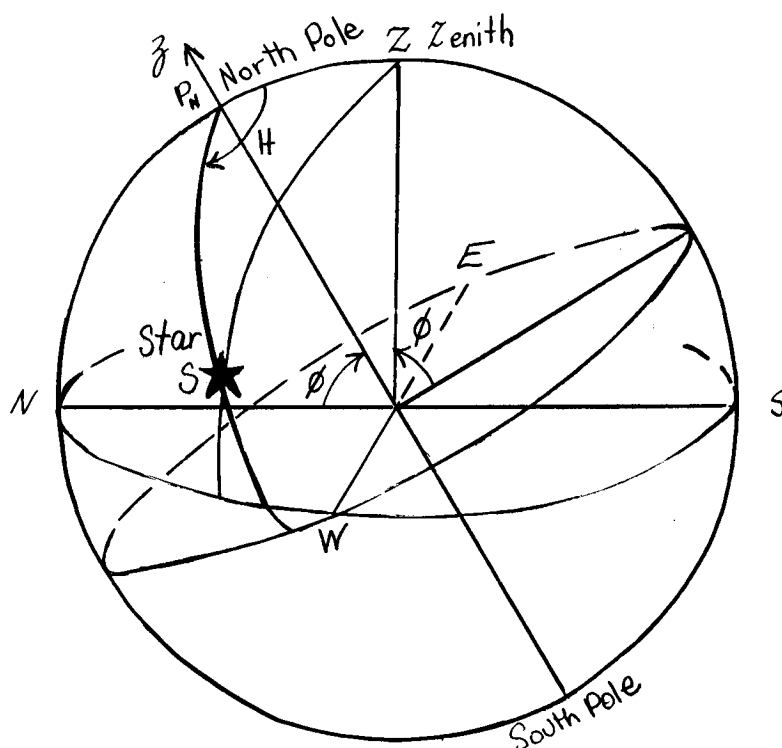
Local terrain also has an effect on the determination of zenith. On a mountain slope inclined upward from east to west, the local vertical will tilt to the east. This is known as local anomaly.

To determine the astronomical longitude and latitude of an observer, we need another coordinate system. Due to the rotation of the Earth, the universe seems to revolve around an axis through the observer and parallel to the Earth's axis. This axis meets the celestial sphere in the north and south celestial poles, and the great circle

Fig. 3

H - Hour angle

PSZ - Navigation
triangle

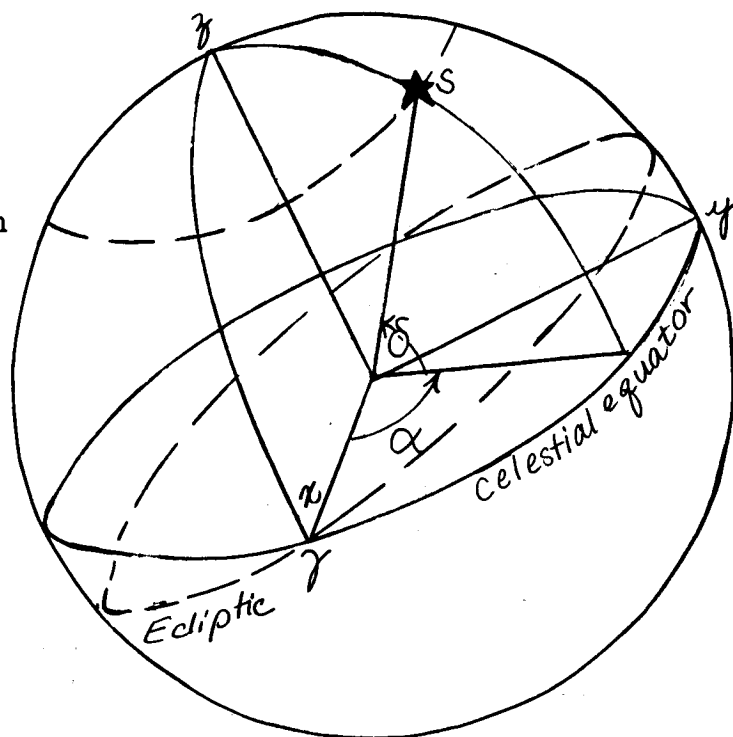


Using the plane of the celestial equator as the base plane, and choosing the vernal equinox, which is the intersecting point of the celestial equator and the ecliptic (the path described by the sun on the celestial sphere), as the zero point, we have the celestial equatorial coordinate system. The position of any star or satellite is measured by the right ascension (angle α) and declination (angle δ) (Fig. 4). Since the axis of the Earth precesses slowly, the equinox is actually moving westward along the ecliptic (Section V). The vernal equinox of 1950 is now taken to be the standard equinox.

Fig. 4

 γ - Vernal equinox

S - Star

 α - Right ascension δ - North declination

Measurements of the declinations and right ascensions of stars are of primary interest to many observatories, notably Yale Observatory, for the purpose of compiling star catalogs. Equatorial coordinates of all planets in the solar system from 1650 to 2050 with respect to the standard equinox were all computed by Eckert, Brouwer and Clemence (Astronomical Papers of American Ephemeris, Vol. XII) around 1950. They are now available in punched card and tape forms.

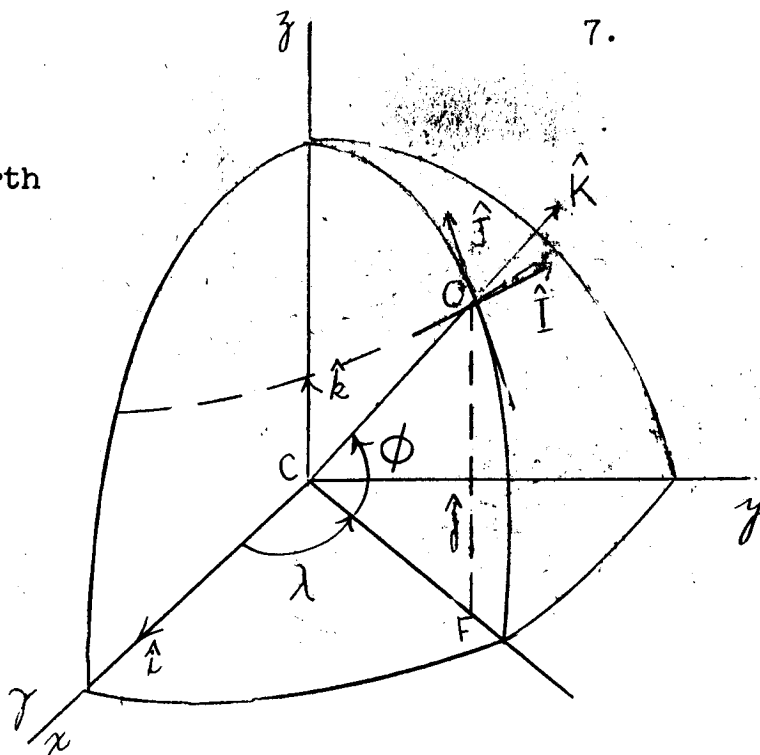
For celestial measurements, the observing station, although it is on the surface of the Earth, may be regarded as being in the center. The movements of stars so observed are known as proper motion. Furthermore, for star observation, we can even shift the station from the center of the Earth to the center of the Sun. If the distance from the Earth to the Sun is taken to be one

astronomical unit and represented graphically by one inch, then the distance to Mercury is 0.4 in.; to Venus 0.7 in.; to Mars 1.52 in.; to Jupiter 5.2 in.; to Saturn 9.5 in.; to Uranus about 20 in.; to Neptune 30 in.; and to Pluto 40 in. But the distance to the nearest star outside the solar system is about four and one-quarter miles (on the graph) away. Thus one is justified in considering the center of the Sun as the observing station. However, for observation inside the solar system, we cannot shift the center arbitrarily.

With the origin at the center of the Earth, an equatorial coordinate system can be used to locate the observation station on the surface of the Earth. Great circles passing through the poles of the Earth are called meridians. The one passing through the vernal equinox is chosen to be the zero hour meridian. The height of the observing station above the equator is measured by the angle ϕ known as astronomical latitude*. The angle λ between the zero hour meridian and the meridian of the station, measured in sidereal hours, is the astronomical longitude.*

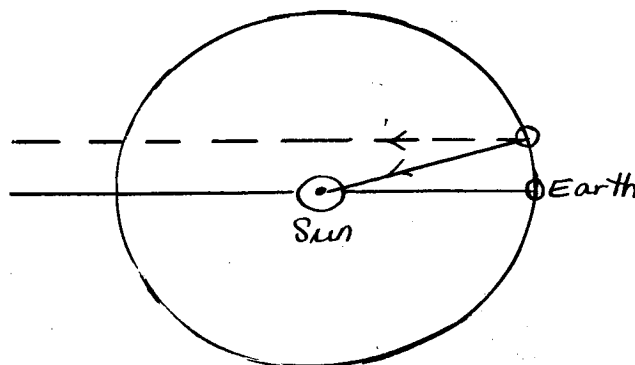
*This should be distinguished from two other systems, geographic and celestial. The zero hour meridian of geographic longitude passes through Greenwich, England. The difference between geographic and astronomical latitudes is shown in Fig. 2. Celestial longitude and latitude are measured along and above the ecliptic respectively.

C - Center of Earth



Adoption of sidereal time leads to the simple rule that the right ascension of an object is equal to the sidereal time at which it transits the meridian.

$$\frac{\text{Sidereal time}}{\text{Solar time}} = \frac{366.2422}{365.2422} = 1.0027... \quad (4)$$



Coordinates with the center of the Earth as their origin will be designated as geocentric; with the center of the Sun as their origin as heliocentric. Conversion of one to the other can easily be done if the geocentric equatorial coordinates of the Sun, or the solar coordinates, are known.

Since the published ephemerides of solar coordinates have the center of the Earth as the origin, whereas the observations are made from the Earth's surface, a slight correction for the parallax is required. In Fig. 5, if the distance CO from the center to the surface of the Earth is expressed in astronomical units, the corrections to be added to the solar coordinates are

$$\begin{aligned}\delta x &= -A \cos \lambda \\ \delta y &= -A \sin \lambda \\ \delta z &= -4266 \times 10^{-8} \sin \phi \\ A &= \overline{CF} = 4266 \times 10^{-8} \cos \phi\end{aligned}\tag{5}$$

Another correction owing to the aberration of light should be noted. Aberration of light is caused by the finite velocity of light and the motion of the observer. When the Earth has a component of motion perpendicular to the line of sight, the light does not reach the Earth along the line joining the Earth and the object, but along the line joining the Earth and the point where the object was at a previous time when the light left it. Thus the observed position of the object is in advance of the computed position given in ephemeris by

$$\text{angle } \alpha = \tan^{-1}(v_E/v_L),$$

where V_L is the speed of light and V_E is the velocity of the Earth perpendicular to the line of sight (Fig. 7).

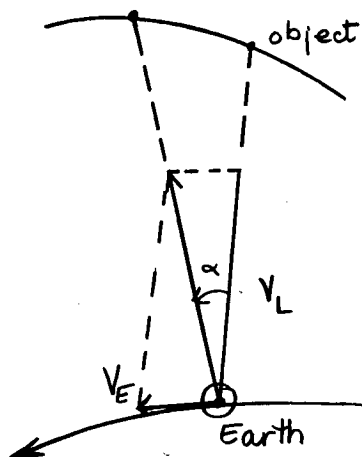


Fig. 7

We now turn to the conversion of local and equatorial coordinates. Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along the geocentric equatorial cartesian coordinates x, y, z , and $\hat{I}, \hat{J}, \hat{K}$ be the unit vectors of the geocentric local cartesian coordinates X, Y, Z . From Fig. 5, it can be shown that

$$\begin{aligned}\hat{I} &= -\sin \lambda \hat{i} + \cos \lambda \hat{j} \\ \hat{K} &= \cos \phi \cos \lambda \hat{i} + \cos \phi \sin \lambda \hat{j} + \sin \phi \hat{k} \\ \hat{J} &= \hat{K} \times \hat{I} \\ &= -\sin \phi \cos \lambda \hat{i} - \sin \phi \sin \lambda \hat{j} + \cos \phi \hat{k}\end{aligned}\tag{6}$$

An object can be located by position vector \bar{r} with different components in two coordinate systems:

$$\begin{aligned}\bar{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ &= X\hat{I} + Y\hat{J} + Z\hat{K}.\end{aligned}$$

Arrange the direction cosines between unit vectors in matrix form:

$$[l_{ij}] = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix}. \quad (6a)$$

Then the transformation of one coordinate to the other can be performed by matrix multiplication*:

$$\text{and} \quad \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ l_{12} & l_{22} & l_{32} \\ l_{13} & l_{23} & l_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Note that the second square matrix is the transpose of the first one, and the inverse of matrix $[l_{ij}]$ equals the transpose** because the direction cosines satisfy the condition

$$\sum_k l_{mk} l_{kn} = \delta_{mn}.$$

*If a_{ij} and b_{ij} are the elements of matrices $[a_{ij}]$ and $[b_{ij}]$ respectively, with i indicating the number of rows and j the number of columns, the product of these two matrices is $[c_{ij}]$, $[a_{ij}] \cdot [b_{ij}] = [c_{ij}]$, with $c_{ij} = \sum_k a_{ik} b_{kj}$. The multiplication rule for Cracovian with $c_{ij} = \sum_k a_{ik} b_{kj}$ has lost ground since the development of electronic computers.

**The transpose of a matrix is obtained by interchanging the rows and columns. The inverse of a matrix $[l_{ij}]$, denoted by $[l_{ij}]^{-1}$, is defined as

$$[l_{ij}] [l_{ij}]^{-1} = [\delta_{ij}]$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

II. Calibration.

Astronomical observations are made with the aid of telescopes. An astronomical telescope must be calibrated before it can be used to locate accurately the position of any object. Artificial satellites which travel at a fast speed around the Earth are difficult to track by optical means. In such a case, radio waves transmitted from the satellite are picked up by a ground antenna, just like optical waves being collected by a telescope. The orientation of the antenna when properly mounted yields the angular position of the satellite. The ground antenna also must be calibrated.

Ground antennae in the MiniTrack system are calibrated against the stars of known positions and an airplane flying overhead. The plane is equipped with a strobe light and a radio transmitter. The radio waves are received by the ground antenna, and the strobe light as well as the stars in the background are recorded by a telescopic camera at the center of the antenna.

Two antennae, one oriented north - south, one east - west, are used to record the direction cosines of the radio transmitter on the airplane. More accurate readings are obtained from the position of the image of the strobe light on the photographic plate relative to the background stars.

Let l and m be the correct values of the two direction cosines as determined from the image, and r_l and r_m be the readings taken by the antennae. Equations used

for calibration are

$$\begin{aligned} l &= r_l + a + bl + cm + dlm + el^2 + fm^2 \\ m &= r_m + a' + b'l + c'm + d'lm + e'l^2 + f'm^2 + g'm^3, \end{aligned} \quad (8)$$

where a, \dots, f and a', \dots, g' are calibration constants. These constants are calculated from a large number of values (over 700) of l, m, r_l and r_m . After the calibration is done, the antennae are used to track the satellite, equipped also with a radio transmitter.

III. Reduction of an Astrographic Plate*.

A photographic plate which records the images of stars and other objects as viewed from an astronomical telescope is called an astrographic plate. The plate is usually centered at a star (point C in Fig. 8) with known declination and right ascension. On the plate, projection of another object (point S) is measured from the center.

Let the plane CL'S' be tangent to the celestial sphere at point C, and let $\hat{\rho}_0, \hat{A}$ and \hat{D} be three mutually perpendicular unit vectors at C, with $\hat{\rho}_0$ normal to the tangent plane and \hat{A} parallel to the xy plane (Fig. 8). We want to determine the position of the star S projected onto the plane.

The direction cosines of the unit vectors $\hat{\rho}_0, \hat{A}, \hat{D}$

*References: Hamburger Sternwarte, Band 5, No. 19.
 W.M. Smart, Spherical Astronomy, Ch. 12.
 Turner, Monthly Notices of the Royal Astronomy Society, v. 54, p. 11 (1893).
 Koenig, Handbuch der Astrophysik, Ch. 6.
 Yale Observatory, v. 9.

and $\hat{\rho}$ which are along the radial direction OS are tabulated below:

	$\hat{\rho}_0$	\hat{A}	\hat{D}	$\hat{\rho}$
x	$\cos \delta_0$	0	$-\sin \delta_0$	$\cos \delta \cos \Delta \alpha$
y	0	1	0	$\cos \delta \sin \Delta \alpha$
z	$\sin \delta_0$	0	$\cos \delta_0$	$\sin \delta$

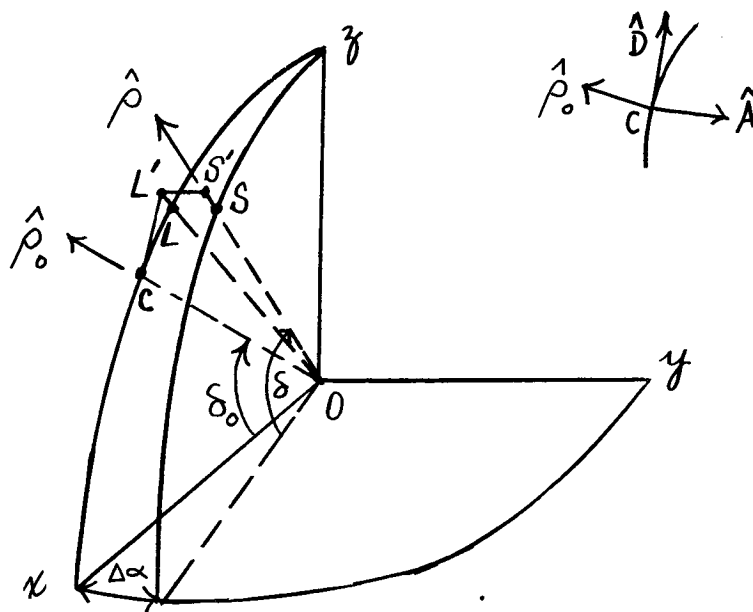


Fig. 8

The position vector

$$\overline{OS'} = \rho \hat{\rho} = \hat{\rho}_0 + \xi \hat{A} + \eta \hat{D}$$

where ξ and η , known as ideal coordinates, are to be

calculated. Note that

$$\begin{aligned}
 \hat{\rho} \cdot \hat{A} \times \hat{D} &= 1 \\
 \rho \hat{\rho} \cdot \hat{\rho}_0 &= 1 \\
 &= \rho (\sin \delta \sin \delta_0 + \cos \delta_0 \cos \delta \cos \Delta \alpha) \\
 &= \rho [\cos \Delta \delta - \cos \delta_0 \cos \delta (1 - \cos \Delta \alpha)]
 \end{aligned}$$

where $\Delta \delta = \delta - \delta_0$,

$$\begin{aligned}
 \rho \hat{\rho} \cdot \hat{A} &= \xi = \rho \cos \delta \sin \Delta \alpha \\
 \rho \hat{\rho} \cdot \hat{D} &= \eta \\
 &= \rho (\sin \delta \cos \delta_0 - \cos \delta \sin \delta_0 \cos \Delta \alpha) \\
 &= \rho [\sin \Delta \delta + \sin \delta_0 \cos \delta (1 - \cos \Delta \alpha)]
 \end{aligned}$$

Thus

$$\begin{aligned}
 \xi &= \cos \delta \sin \Delta \alpha / D \\
 \eta &= [\sin \Delta \delta + \sin \delta_0 \cos \delta (1 - \cos \Delta \alpha)] / D \\
 D &= \cos \Delta \delta - \cos \delta_0 \cos \delta (1 - \cos \Delta \alpha). \quad (9)
 \end{aligned}$$

Now consider the tangent plane to be the photographic plate. The projected position of star S on the plate relative to point C can be deduced from equations (9) if δ and $\Delta \alpha$ are given.

On the other hand, the inverse procedure of determining the right ascension and declination of S' from an astrographic plate is not so simple. First of all, because of the centering and orientation error of the plate, the refraction of light in the atmosphere, and optical distortion of the instrument, the recorded position of S on the plate with coordinates X, Y is not the same as the ideal coordinates ξ, η on the tangent plane (Fig. 9). Next, even if the ideal coordinates are known, calculation of $\Delta \alpha$ and $\Delta \delta$ from equation (9) is not straightforward. We shall first discuss various sources of errors on the

plate.

The misalignment of axes and center point of the photographic plate will shift the coordinates of S' according to (Fig. 9)

$$\begin{aligned}\xi &= a + bX + cY \\ \eta &= a' + b'X + c'Y,\end{aligned}\quad (10)$$

where a, \dots, c' are constants.

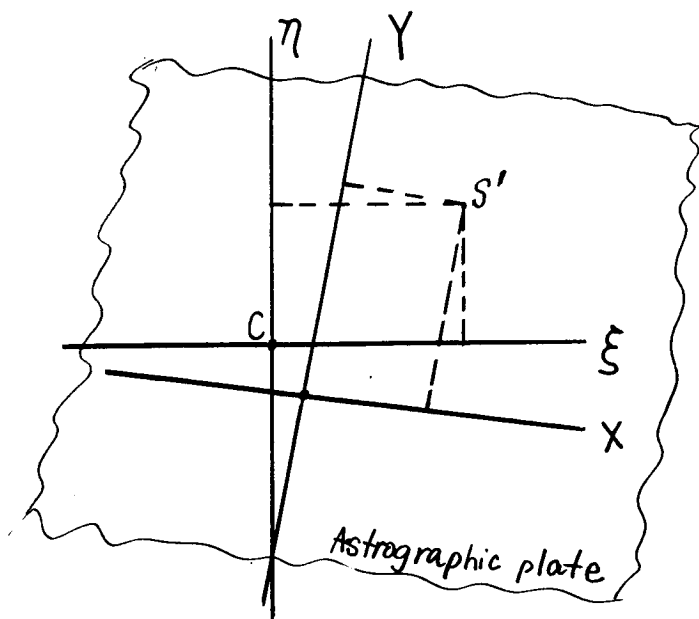


Fig. 9

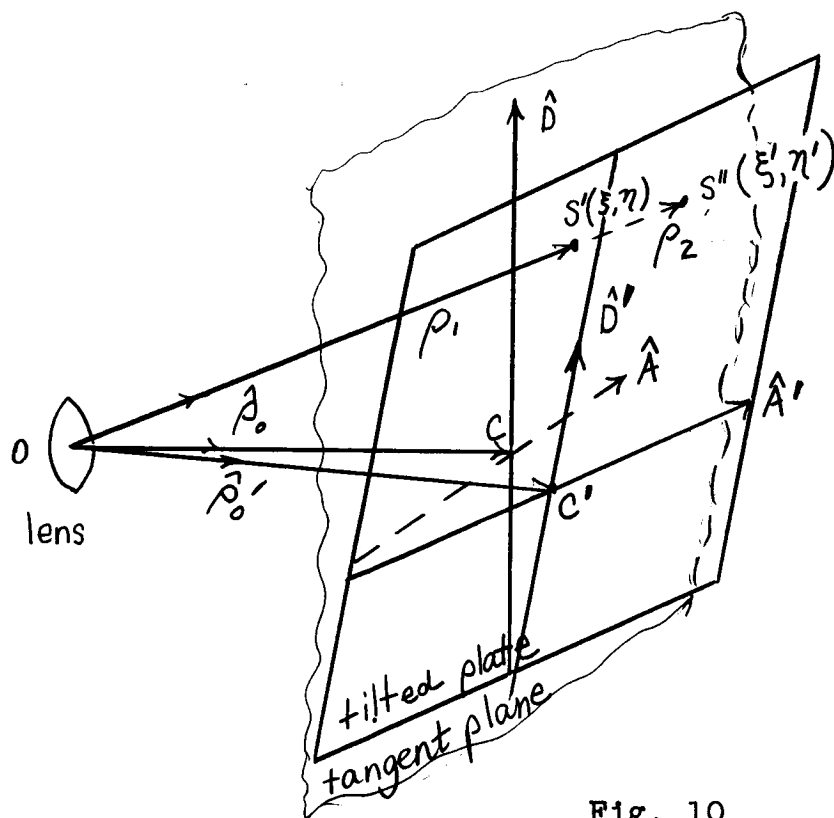
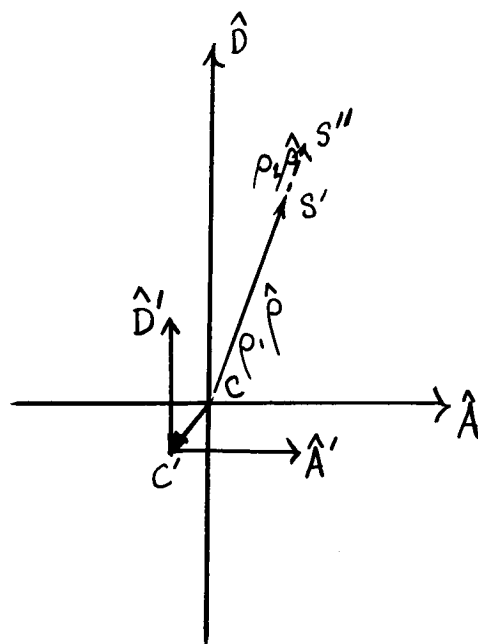


Fig. 10



The tilted plate, when centered at the star C, will shift the line of sight from OC to OC' (Fig. 10). The unit vectors \hat{A}' and \hat{D}' along the axes on the tilted plate are rotated by a small amount about the unit vectors \hat{A} and \hat{D} on the tangent plane. We thus have

$$\begin{aligned}\hat{\rho}_0' &= \hat{\rho}_0 - p\hat{A} - q\hat{D} \\ \hat{A}' &= \hat{A} + p\hat{\rho}_0 \\ \hat{D}' &= \hat{D} + q\hat{\rho}_0\end{aligned}$$

where p and q are magnitudes of rotations. Noting that

$$\begin{aligned}\rho_1 \hat{\rho} \cdot \hat{A} &= \xi \\ \rho_1 \hat{\rho} \cdot \hat{D} &= \eta \\ \rho_2 \hat{\rho} \cdot \hat{A}' &= \xi' \\ \rho_2 \hat{\rho} \cdot \hat{D}' &= \eta',\end{aligned}$$

and setting

$$\rho_2 \hat{\rho} \cdot \hat{\rho}_0' = 1,$$

we have

$$\begin{aligned}\xi' &= \rho_2 \hat{\rho} \cdot (\hat{A} + p\hat{\rho}_0) = \rho_2 \left(\frac{\xi}{\rho_1} + p/\rho_1 \right) \\ \eta' &= \rho_2 \hat{\rho} \cdot (\hat{D} + q\hat{\rho}_0) = \rho_2 \left(\frac{\eta}{\rho_1} + q/\rho_1 \right) \\ 1 &= \rho_2 \hat{\rho} \cdot (\hat{\rho}_0 - p\hat{A} - q\hat{D}) = \frac{\rho_2}{\rho_1} (1 - p\xi - q\eta).\end{aligned}$$

Finally, where higher powers of p and q are neglected,

$$\begin{cases} \rho_2/\rho_1 = 1 + p\xi + q\eta \\ \xi' - \xi = p + p\xi^2 + q\xi\eta \\ \eta' - \eta = q + p\xi\eta + q\eta^2. \end{cases} \quad (11)$$

This result shows that the error due to the tilt of the astrographic plate depends on the magnitude of tilting and the quadratic power of the coordinates.

Refraction of light in the atmosphere causes the images to be displaced systematically toward the zenith. The optical distortion in the instrument is related to the cubic power of the distance from center C on the plate. All in all, we can compensate these errors by letting

$$\begin{aligned}\xi &= a + bX + cY + dXY + eX^2 + fX(X^2 + Y^2) \\ \eta &= a' + b'X + c'Y + d'XY + e'Y^2 + f'Y(X^2 + Y^2),\end{aligned}\quad (12)$$

where a, \dots, f and a', \dots, f' are called plate constants. These constants are calibrated from a number of stars with known declinations and right ascensions, from which the ideal coordinates can be computed from equation (9). Their corresponding coordinates (X, Y) can be directly measured from the astrographic plate. Inserting these sets of values into equation (12), we then can calculate the plate constants.

Once the plate constants are fixed, the same equation will be used to compute the ideal coordinates of any unknown object recorded on the same plate. The right ascension and declination of the object are then calculated by applying an iterative method to equation (9).

Equations (9) are rearranged below:

$$\begin{aligned}A = \sin \Delta \delta &= \xi D - \sin \delta_0 \cos \delta (1 - \cos \Delta \alpha) \\ \cos \Delta \delta &= \sqrt{1 - \sin^2 \Delta \delta} \\ \cos \delta &= \cos \delta_0 \cos \Delta \delta - \sin \delta_0 \sin \Delta \delta \\ B = \sin \Delta \alpha &= \xi / D \cos \delta \\ \cos \Delta \alpha &= \sqrt{1 - \sin^2 \Delta \alpha}\end{aligned}\quad (9a)$$

Initially, set $\Delta\alpha$ and $\Delta\delta$ equal to zero and compute A, B and all other quantities in order. Call the results of the first approximation A_1, B_1 , etc., repeating the process until the nth approximation with

$$(A_n - A_{n-1})^2 + (B_n - B_{n-1})^2 \leq \epsilon$$

is reached. The ϵ is a preassigned small tolerance, usually 10^{-7} .

Astrographic plates of the entire celestial sphere have been made by observatories over the world under the project initiated about 1890. Each plate covers two degrees in latitude and in longitude. Every quarter of each plate is overlapped with another plate so that the entire sphere is covered twice. Since plate constants are different from station to station, the equations used to convert plate coordinates to ideal coordinates also vary.

In the Publications of the Hamburger Sternwarte, Band 5, are given formulas and tables for the uniform reduction of plate measures for all the zones of the Astrographic Catalogues and the conversion from the rectangular ideal coordinates to right ascension and declination. Use is made of the right spherical triangle ZLS and Napier's Rules, namely

$$\sin = \cos \cdot \cos (\text{opposites}) = \tan \cdot \tan (\text{adjacents}).$$

In the Hamburg notation,

$$\begin{aligned}
 \eta &= \tan(d - \delta_0), \quad \xi \cos(d - \delta_0) = \tan \nu = \tan LS \\
 \sin \delta &= \sin d \cos \nu \\
 \sin(90 - \Delta\alpha) &= \tan \delta \tan(90 - d) \\
 \tan \delta &= \tan d \cos \Delta\alpha \\
 \cos \delta &= \cos d \cos \nu \sec \Delta\alpha \\
 \sin(90 - d) &= \tan \nu \tan(90 - \Delta\alpha) \\
 \tan \Delta\alpha &= \tan \nu \sec d = \xi \cos(d - \delta_0) \sec d \\
 &= \xi(1 - N) \sec d \\
 d &= \delta_0 + \eta - T(\eta) \\
 \sin(d - \delta) &= \sin d \cos \delta - \sin \delta \cos d \\
 &= \sin d \cos d \cos \nu [\sec \Delta\alpha - 1] \\
 &= \frac{1}{2} \sin 2d (1 - \text{vers } \nu) [\sec \Delta\alpha - 1] \\
 \delta &= d - D \sin^2 D
 \end{aligned}$$

For the $2^\circ \times 2^\circ$ plates of the Astrographic Catalogue, tables give values of N and D as a function of η , and another table gives the conversion from $\tan \Delta\alpha$ to $\Delta\alpha$.

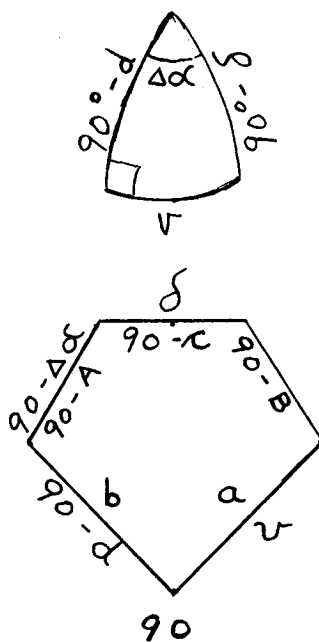


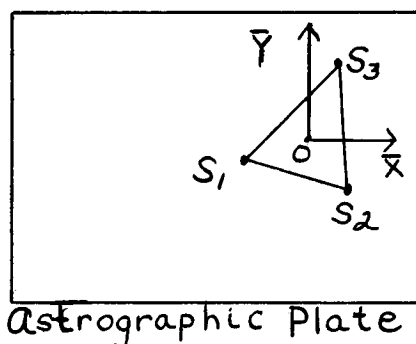
Fig. 11

IV. Dependences.

When the unknown object is surrounded by a cluster of stars with known positions, the ideal coordinates of the object can be computed from those of the stars. Let \bar{r} be the position vector of the unknown object and \bar{r}_1 , \bar{r}_2 , and \bar{r}_3 be those of three stars, S_1 , S_2 , and S_3 (Fig. 12). Then

$$\bar{r} = D_1 \bar{r}_1 + D_2 \bar{r}_2 + D_3 \bar{r}_3, \quad (13)$$

where the D's are coefficients. On the astrographic plate, taking the image of the unknown object as the origin, we have three equations to determine the coefficients:



$$\begin{aligned} D_1 X_1 + D_2 X_2 + D_3 X_3 &= 0 \\ D_1 Y_1 + D_2 Y_2 + D_3 Y_3 &= 0 \\ D_1 + D_2 + D_3 &= 1 \text{ (normalized).} \end{aligned}$$

Fig. 12

The ideal coordinates of the object are then given by

$$\begin{aligned} \xi &= D_1 \xi_1 + D_2 \xi_2 + D_3 \xi_3 \\ \eta &= D_1 \eta_1 + D_2 \eta_2 + D_3 \eta_3. \end{aligned}$$

This method has the advantage of avoiding the calibration of plate constants*. However, it does not yield good results when there are no known stars closely surrounding the object.

*Arend: Bulletin of Astronomy of Brussels, v. 1, pp. 124, 199.

V. Precession.

As a result of the attraction of the Sun and Moon on the Earth's bulges near its equator, the Earth precesses very slowly (approximately one complete rotation in 26,000 years) about the axis perpendicular to the ecliptic. This ~~precession will then cause the equinox,~~ which is the intersection of celestial equator and ecliptic, to move. The declination and right ascension in the celestial equatorial coordinate system will thus change from time to time because of the changing zero point. These changes will now be computed.

Fig. 13 shows that the equator precesses about the axis \hat{K} , perpendicular to the ecliptic, with angular rotation $\Delta\theta$. The direction cosines of the unit vectors \hat{A} , \hat{D} , \hat{K} , \hat{S} , referring to the celestial equatorial coordinates, are tabulated below.

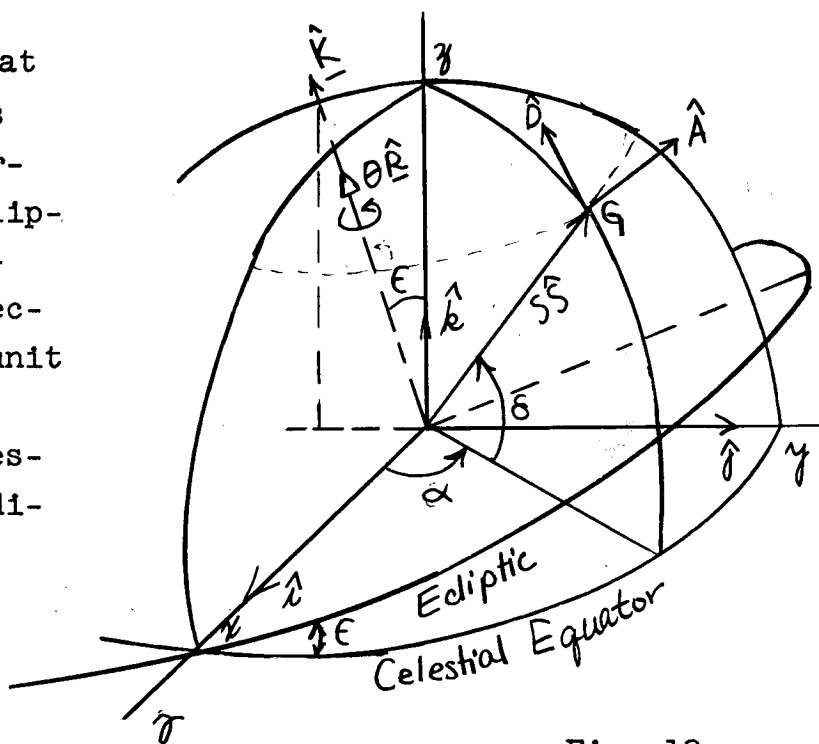


Fig. 13

	\hat{K}	\hat{S}	\hat{A}	\hat{D}	$\Delta \hat{S}$
\hat{i}	0	$\cos \delta \cos \alpha$	$\sin \alpha$	$-\sin \delta \cos \alpha$	$\Delta \theta (-\sin \epsilon \sin \delta$ $- \cos \epsilon \cos \delta \sin \alpha)$
\hat{j}	$-\sin \epsilon$	$\cos \delta \sin \alpha$	$\cos \alpha$	$-\sin \delta \sin \alpha$	$\Delta \theta \cos \epsilon \cos \delta \cos \alpha$
\hat{k}	$\cos \epsilon$	$\sin \delta$	0	$\cos \delta$	$\Delta \theta \sin \epsilon \cos \delta \cos \alpha$

The last column in the table lists the components of the change of unit vector \hat{S} due to precession with

$$\Delta \hat{S} = \Delta \theta \hat{K} \times \hat{S}.$$

The components of the same change along \hat{A} and \hat{D} directions yield the increments of α and δ of object C with

$$\begin{aligned} \Delta \hat{S} \cdot \hat{A} &= \Delta \theta (\sin \epsilon \sin \delta \sin \alpha + \cos \epsilon \cos \delta) = \cos \delta \Delta \alpha \\ \Delta \hat{S} \cdot \hat{D} &= \Delta \theta (\sin \epsilon \cos \alpha) = \Delta \delta, \end{aligned}$$

and

$$\begin{aligned} \frac{\Delta \alpha}{\Delta t} &= \frac{\Delta \theta}{\Delta t} (\cos \epsilon + \sin \epsilon \sin \alpha \tan \delta) \\ \frac{\Delta \delta}{\Delta t} &= \frac{\Delta \theta}{\Delta t} (\sin \epsilon \cos \alpha). \end{aligned}$$

In the limit, as time t approaches zero,

$$\begin{aligned} \frac{d\alpha}{dt} &= m + n \sin \alpha \tan \delta \\ \frac{d\delta}{dt} &= n \cos \alpha, \end{aligned} \tag{14}$$

where m and n are known constants determined from the rate of precession and obliquity ϵ . In terms of second / year,

$$\begin{aligned} m &= +3.07327 + 0.0000186(t - 1950) \\ n &= +1.33617 - 0.0000057(t - 1950). \end{aligned}$$

VI. Astronomical Refraction*.

Because of refraction in the atmosphere, the light path from a star to an observer is a curved one. The star appears to the observer at a position different from its actual one, being shifted toward the zenith.

Let the altitude of a star S be $\pi/2 - \zeta$, and the index of refraction of the atmosphere be μ . Consider the atmosphere to be composed of many layers as shown in Fig. 14.

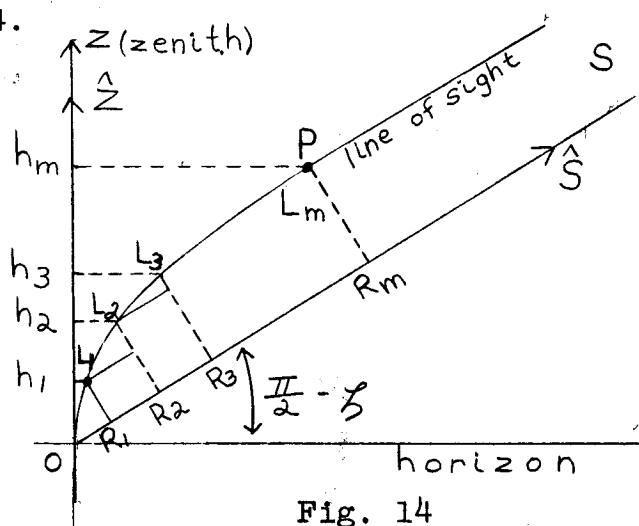


Fig. 14

In layer 1,

$$\begin{aligned} \angle L_1 O R_1 &= (\mu_1 - 1) \tan \zeta \\ L_1 R_1 &= O R_1 \cdot \tan(\angle L_1 O R_1) \\ &\approx (\mu_1 - 1) \tan \zeta (h_1 - h_0) \sec \zeta. \end{aligned}$$

In layer 2,

$$\begin{aligned} L_2 R_2 &= (\mu_2 - 1) \tan \zeta \cdot (h_2 - h_1) \sec \zeta + L_1 R_1 \\ &= \tan \zeta [(\mu_2 - 1)(h_2 - h_1) + (\mu_1 - 1)(h_1 - h_0)] \sec \zeta. \end{aligned}$$

*Smart: Spherical Astronomy, p. 60.

Finally, at point P,

$$L_m R_m = \tan \zeta \sum [(\mu_m - 1)(h_m - h_{m-1})] \cdot \frac{R_m R_0}{h_m h_0}.$$

If \hat{W} is the unit vector perpendicular to the ZS plane, then $\hat{W} \times \hat{S}$ is the unit vector along the PR_m direction.

$$\begin{aligned}\hat{W} &= \frac{\hat{S} \times \hat{Z}}{\sin \zeta} \\ \hat{W} \times \hat{S} &= \frac{\hat{S} \times \hat{Z}}{\sin \zeta} \times \hat{S} = \frac{\hat{Z} - (\hat{S} \cdot \hat{Z})\hat{S}}{\sin \zeta} = \csc \zeta \hat{Z} - \cot \zeta \hat{S}\end{aligned}$$

Then the position vector of P is

$$\begin{aligned}\vec{OP} &= R_m R_0 \hat{S} + L_m R_m (\hat{W} \times \hat{S}) \\ &= R_m R_0 \left[\hat{S} + \frac{\sum (\mu_m - 1)(h_m - h_{m-1})}{h_m h_0} (\sec \zeta \hat{Z} - \hat{S}) \right],\end{aligned}\tag{15}$$

from which the angle POS can be computed. Several typical values of the angle POS at 45° altitude are given below.

$h(\text{km})$	0	2	4	6	8	10	20
$\angle \text{POS}(\text{sec})$	58	53	48	44	40	36	23

VII. Occultations.

The hiding of one object in the sky by another, especially when ~~a moon passes in front of a star or a~~ planet, is known as occultation.

Imagine one standing on a star looking at the Earth along the z axis (Fig. 15), seeing a moon sweep in front of the Earth. Take the meridian directly facing the star

An observer on the surface of the Earth with local hour h (longitude from central meridian) and latitude ϕ' will have coordinates*

$$\begin{aligned}\xi &= \rho \cos \phi' \sin h \\ \eta &= \rho (\sin \phi' \cos \delta - \cos \phi' \sin \delta \cos h)\end{aligned}\quad (17)$$

on the $x - y$ plane. If the radius of the Earth ρ is taken to be unity, the moon's radius is then 0.2725. When the observer is on the periphery of the moon,

$$(\xi - x)^2 + (\eta - y)^2 = (0.2725)^2. \quad (18)$$

Occultation can be utilized in two ways. If the observer's position (longitude and latitude) is exactly known, he will be on the rim of the moon whenever the above equation is satisfied. Thus one can determine when the occultation will take place. On the other hand, from the observed and calculated occultation, the position of the observing station can be determined.

*The derivation of (17) is the same as that of (9). The x and y axes in Fig. 15 correspond to the \hat{A} and \hat{D} vectors on the tangent plane of Fig. 8. The angles h , δ and ϕ' (not shown) on Fig. 15 correspond to angles $\Delta\alpha$, δ_0 and δ on Fig. 8 respectively.

as the central meridian which has hour angle H_0 measured from Greenwich zero. At time T_0 , the center of the moon passes through the central meridian at the point $(0, -Y_0)$ on the $x - y$ plane. After an increment of time Δt , the center of the moon will be at

$$\begin{aligned} x &= \dot{x} \Delta t \\ y &= Y_0 + \dot{y} \Delta t, \end{aligned} \quad (16)$$

where \dot{x} and \dot{y} are the rates of change of the moon along the x and y directions. Given H_0 , T_0 , Y_0 , \dot{x} , and \dot{y} , one can compute the position of the center of the moon at any time.

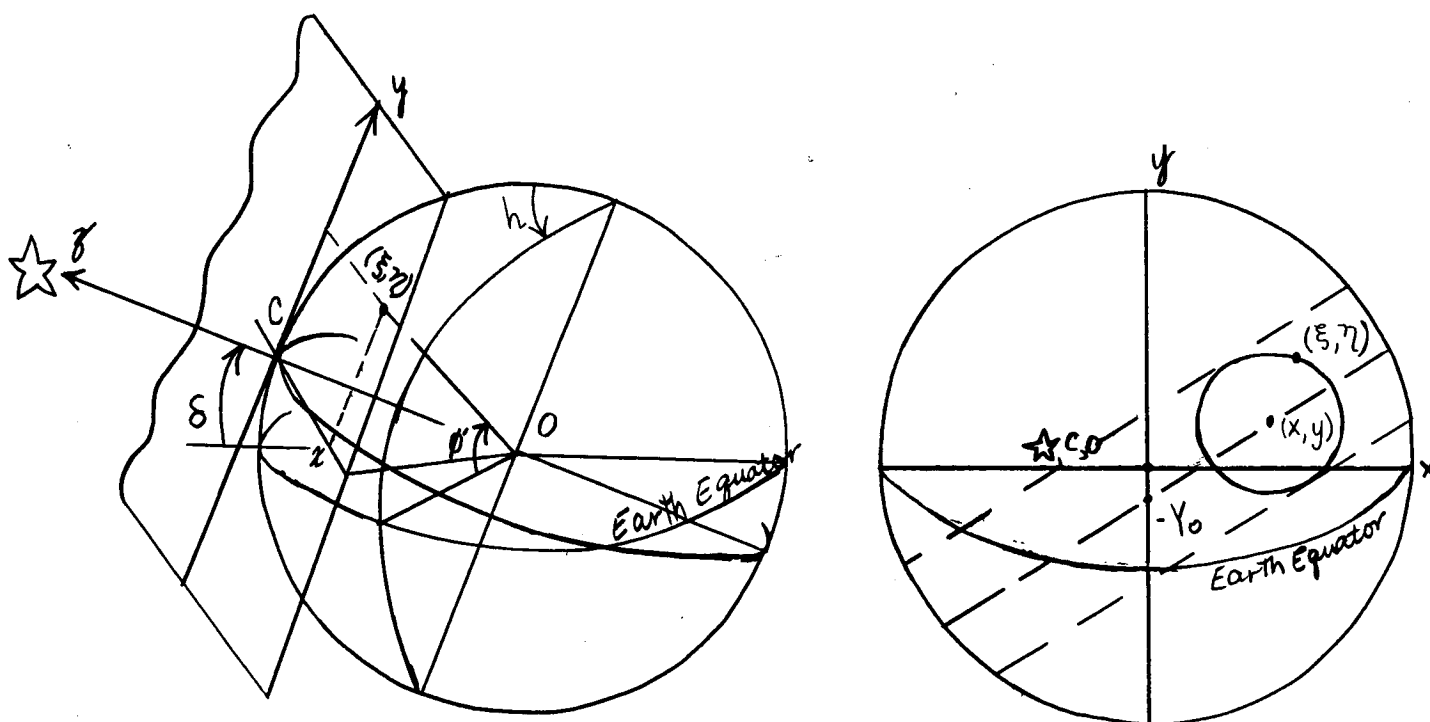


Fig. 15

Orbit Determination
by
Professor Paul Herget

Orbit Determination.

I. The Method of Laplace.

A. The Preliminary Orbit.

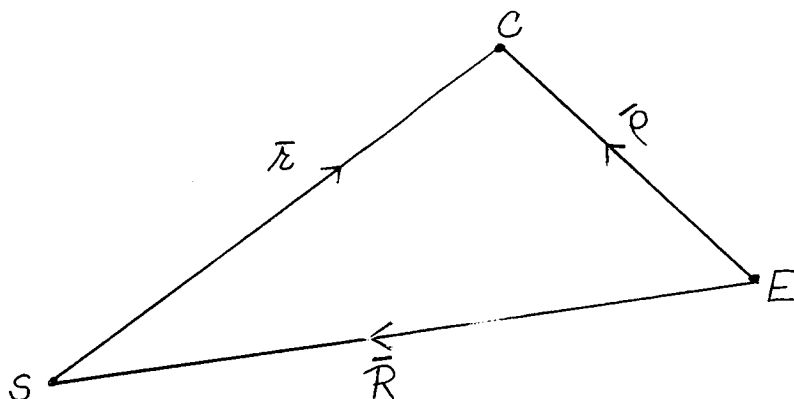


Figure 1.

S: the position of the sun, origin of co-ordinate system

C: the position of the newly discovered object

E: the position of the Earth

Suppose there are three observations of a new object made one day apart. That which is observed is

$$\hat{\rho}_1 = (\cos \alpha_1 \cos \delta_1, \sin \alpha_1 \cos \delta_1, \sin \delta_1) \quad 1 = 1, 2, 3.$$

The basic equation connecting the three position vectors in Figure 1 is

$$(1) \quad \bar{r} = \rho \hat{\rho} - \bar{R} \quad , \quad \text{where } \rho \hat{\rho} = \bar{\rho} \quad , \quad \rho = |\bar{\rho}|$$

Differentiating,

$$(2) \quad \frac{d\bar{r}}{dt} = \frac{d\rho}{dt} \hat{\rho} + \rho \frac{d\hat{\rho}}{dt} - \frac{d\bar{R}}{dt}$$

$$(3) \quad \frac{d^2\bar{r}}{dt^2} = \frac{d^2\rho}{dt^2} \hat{\rho} + 2 \frac{d\rho}{dt} \frac{d\hat{\rho}}{dt} + \rho \frac{d^2\hat{\rho}}{dt^2} - \frac{d^2\bar{R}}{dt^2}$$

$$= - \frac{\bar{r}}{r^3} = - \frac{(\rho \hat{\rho} - \bar{R})}{r^3}$$

Dotting Equation (3) by $\hat{\rho} \times \frac{d\hat{\rho}}{dt}$, we get

$$(4) \quad \rho \left[\hat{\rho} \times \frac{d\rho}{dt} \cdot \frac{d^2\hat{\rho}}{dt^2} \right]$$

$$= \left[\hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \frac{d^2\bar{R}}{dt^2} \right] + \left[\hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \bar{R} \right] \frac{1}{r^3}$$

$$\text{where } \hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \hat{\rho} = 0 = \hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \frac{d\hat{\rho}}{dt}$$

Dividing Equation (4) by $\left[\hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \frac{d^2\hat{\rho}}{dt^2} \right]$, we have

an equation of the form

$$(5) \quad \rho = A + \frac{B}{r^3}$$

From the geometry, it is also true that

$$(6) \quad r^2 = \rho^2 - 2[\hat{\rho} \cdot \bar{R}] \rho + R^2$$

To find A and B in Equation (5), first consider the Taylor series expansion for $\hat{\rho}$.

$$(7) \quad \hat{\rho} = \hat{\rho}_0 + \Delta t \hat{\rho}_0' + \frac{(\Delta t)^2}{2!} \hat{\rho}_0'' + \dots$$

If we truncate the series at this point and let the origin of time, t_0 , be the time of the second observation, we can state the following.

$$W_0' = \frac{(t_3 - t_0) (W,1) - (t_1 - t_0) (W,3)}{t_3 - t_1}$$

$$\frac{1}{2} W_0'' = \frac{(W,3) - (W,1)}{t_3 - t_1}$$

where $(W,1) = \frac{W_1 - W_0}{t_1 - t_0}$

$$(W,3) = \frac{W_3 - W_0}{t_3 - t_0}$$

and W_1 stands for each component of $\hat{\rho}_1$ taken separately.

W_0' and W_0'' , therefore, give the components of

$$\frac{d\hat{\rho}_0}{dt} \quad \text{and} \quad \frac{d^2\hat{\rho}_0}{dt^2}$$

The other quantities needed to calculate the values of A and B are \bar{R} and $\frac{d^2\bar{R}}{dt^2}$.

\bar{R} can be found in the American Ephemeris. Topocentric parallax corrections may have to be applied to this value of \bar{R} .

For short intervals of time the relation

$$\frac{d^2\bar{R}}{dt^2} = -\frac{\bar{R}}{R^3} \quad \text{is sufficiently accurate.}$$

With this information, A and B in Equation (5) can be found. Then some iteration method can be applied to Equations (5) and (6) until ρ and r converge. A possible starting point might be to set $\rho = A$ and solve for r in (6). This value of r gives a new value for ρ from (5), etc.

Once ρ_0 is found, \bar{r}_0 is known.

Equation (2) gives the value for $\frac{d\bar{r}_0}{dt} = \bar{v}_0$.

$\frac{d\bar{R}}{dt}$ is found in the American Ephemeris.

(Numerical example of this method found in Herget, p. 23 and p. 44).

\bar{r}_0 and \bar{v}_0 are the constants of integration needed to determine the orbit. If necessary, the elements of the orbit can be found from these. (Herget, p. 47)

B. Some Difficulties Which May Be Encountered.

1. Extraneous Solutions.

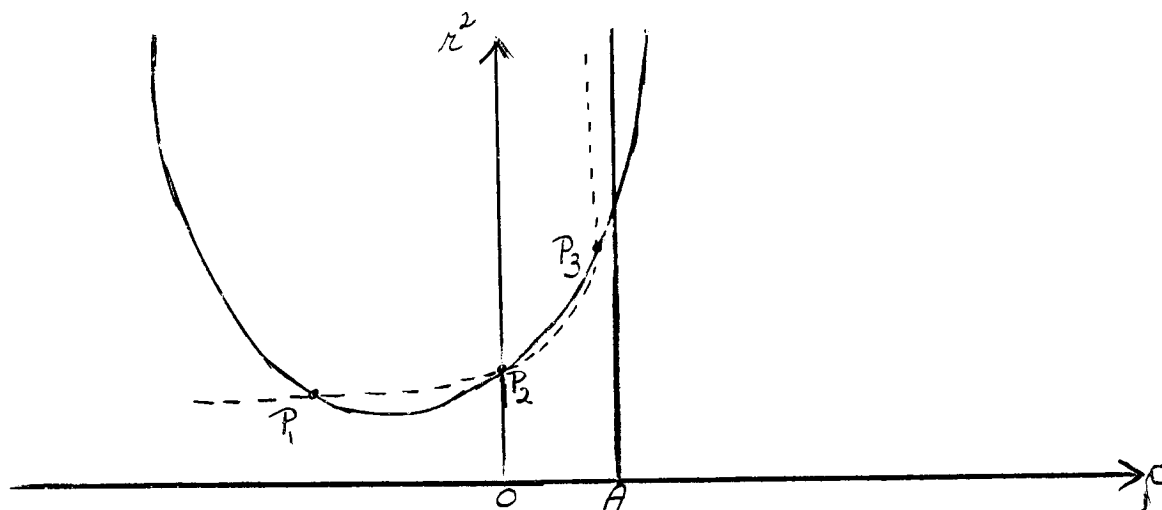


Figure 2.

The dotted curve represents the graph of Equation (5); the solid curve, that of Equation (6).

P_1 is the fictitious solution $\rho < 0$.

P_2 is the position of the observer, which is also a solution to the equations.

P_3 is the real solution.

2. The solutions made for $\hat{\rho}_0'$ and $\hat{\rho}_0''$ are distorted by truncating the Taylor series (7) after the second order.

3. Zero Divisors and Small Divisors (Herget, pp. 38f)

In equation (4), we are dividing by the quantity

$\hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \frac{d^2\hat{\rho}}{dt^2}$. If this quantity is zero or nearly zero, we do not have a determinant equation.

The derivative of a unit vector is always perpendicular to the unit vector. Therefore

$$\frac{d\hat{\rho}}{dt} \perp \hat{\rho} .$$

If $\frac{d^2\hat{\rho}}{dt^2}$ is directed toward the observer, then

$$\hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \frac{d^2\hat{\rho}}{dt^2} = 0 .$$

If the three observations lie

in a plane containing the earth, then $\hat{\rho}$ is moving along a great circle on the celestial sphere. This situation implies that $\frac{d^2\hat{\rho}}{dt^2}$ does not have a component out of

the plane determined by $\hat{\rho}$ and $\frac{d\hat{\rho}}{dt}$. Again $\hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \frac{d^2\hat{\rho}}{dt^2} = 0$.

If r varies in such a way that at one time $|\vec{r}| > |\vec{R}|$ and at some time later $|\vec{r}| < |\vec{R}|$, then the path of the object on the sky will show an inflection point as the sun "moves" from the convex to the concave side of the path. At this inflection point, there is no solution to the equations.

If the path of the object exhibits relatively slow curvature, that is, if the deviation of the path from a great circle is small, then

$$\hat{\rho} \times \frac{d\hat{\rho}}{dt} \cdot \frac{d^2\hat{\rho}}{dt^2} \text{ is very small, proportional to the}$$

area of the spherical triangle joining the points of the three observations. Dividing by a small quantity will inject serious errors into the solution.

One could apply L'Hôpital's Rule from the Differential Calculus to Equation (4), but practically speaking this is not usually done.

If the three observations lie on a great circle, then the three observations are not linearly independent. One might use these observations to determine an orbit needing only four arbitrary constants.

The difficulty of slow curvature may be avoided by allowing a longer time interval between observations. However, as Δt increases, the rapid convergence of the Taylor series (7) becomes challenged.

C. Another Approach to the Problem.

Consider the following equation expressing \bar{r} as a vector sum of \bar{r}_0 and $\frac{d\bar{r}_0}{dt} = \bar{v}_0$.

$$(8) \quad \bar{r}_1 = f_1 \bar{r}_0 + g_1 \bar{v}_0$$

$$(8a) \quad \rho_1 \hat{\rho}_1 - \bar{R}_1 = f_1 (\rho_0 \hat{\rho}_0 - \bar{R}_0) + g_1 \bar{v}_0 \quad \left[\text{Using Eq. (1)} \right]$$

The following set of mutually orthogonal vectors are available from the observations.

$$\begin{aligned}\hat{\rho} &= (\cos\delta \cos\alpha, \cos\delta \sin\alpha, \sin\delta) \\ \hat{A} &= (-\sin\alpha, \cos\alpha, 0) \\ \hat{D} &= (-\sin\delta \cos\alpha, -\sin\delta \sin\alpha, +\cos\delta)\end{aligned}$$

If we dot Equation (8a) first by \hat{A}_1 and then by \hat{D}_1 and re-arrange terms, we get the following two equations.

$$(9) \quad f_1 [\hat{\rho}_0 \cdot \hat{A}_1] \rho_0 + g_1 \hat{A}_1 \cdot \bar{v}_0 = f_1 [\bar{R}_0 \cdot \hat{A}_1] - [\bar{R}_1 \cdot \hat{A}_1]$$

$$(10) \quad f_1 [\hat{\rho}_0 \cdot \hat{D}_1] \rho_0 + g_1 \hat{D}_1 \cdot \bar{v}_0 = f_1 [\bar{R}_0 \cdot \hat{D}_1] - [\bar{R}_1 \cdot \hat{D}_1]$$

Each of these two equations contains four unknowns, ρ_0 and \bar{v}_0 . If we have two observations, (9) and (10) give us four equations in four unknowns.

$$\begin{aligned}f_1 &= 1 - \frac{(\Delta t)^2}{2!} \frac{1}{r_0^3} + \frac{(\Delta t)^3}{2} \frac{r_0'}{r_0^4} + \dots \\ g_1 &= \Delta t \left[1 - \frac{(\Delta t)^2}{3!} \frac{1}{r_0^3} + \dots \right]\end{aligned}$$

f_1 and g_1 are obtained from the Taylor expansion of \underline{r} about \underline{r}_0 .

$$\bar{r} = \bar{r}_0 + \Delta t \dot{\bar{r}}_0 + \frac{(\Delta t)^2}{2!} \ddot{\bar{r}}_0 + \frac{(\Delta t)^3}{3!} \dddot{\bar{r}}_0 + \dots$$

Recalling $\ddot{\bar{r}}_0 = -\frac{\bar{r}_0}{r_0^3}$

$$\dddot{\bar{r}}_0 = \frac{d}{dt} \left(-\frac{\bar{r}_0}{r_0^3} \right)_{t=t_0} = 3 \frac{\dot{r}_0}{r_0^4} \bar{r}_0 - \frac{\ddot{\bar{r}}_0}{r_0^3}$$

A possible method of solving for these quantities is to guess an r_0 , giving an initial f_1 and g_1 . Substitution of these values into equations (9) and (10) yield first approximations to ρ_0 and \bar{v}_0 . From (1), ρ_0 implies \bar{r}_0 . These values of \bar{r}_0 and \bar{v}_0 ($\bar{v}_0 \equiv \dot{\bar{r}}_0$) can be used to find better approximations to f_1 and g_1 .

There is a distinction to be made between the method described in Section A and that in Section C. In (A), the dynamical conditions of the problem were exactly satisfied [Equation (3)], while the geometrical conditions employing the observations were only approximately satisfied in the Taylor series (7) for $\hat{\rho}$.

In the second approach, f_1 and g_1 , which represent the dynamical conditions, are approximated, while the geometry at the observations is exactly represented in (9) and (10).

D. Method of Solution When ρ and $\hat{\rho}$ Are Known.

With the use of radar, it is now possible to determine ρ as well as $\hat{\rho}$. As an illustration of the method used to determine \bar{r}_0 and \bar{v}_0 from the complete vector $\bar{\rho}$, we shall consider observations made by the Bermuda radar equipment used in the Mercury Project.

There is a 20 second interval after the second stage of the Atlas releases the capsule when it must be decided whether or not the capsule can continue in the orbit it now has. Observations are made every 0.1

second during this interval, giving a total of 200 observations. The epoch of time, t_0 , is taken to be the half-way point of the total 20 second interval.

The equation used to relate observations to unknowns is:

$$(11) \quad \left[1 - \frac{(\Delta t)^2}{2 r_0^3} + \frac{(\Delta t)^3}{2 r_0^5} \bar{r}_0 \cdot \bar{v}_0 \right] \bar{r}_0 + \Delta t \left[1 - \frac{(\Delta t)^2}{6 r_0^3} + \frac{(\Delta t)^3}{4 r_0^5} (\bar{r}_0 \cdot \bar{v}_0) \right] \bar{v}_0 = \bar{R}_\oplus + \bar{\rho}$$

\bar{R}_\oplus is the vector from the center of the Earth to the observer. $\bar{\rho}$ is the vector from the observer to the capsule (the observed quantity).

The local coordinate system of the observer is changing slowly during this interval of time. Since $\bar{\rho}$ is measured relative to the local coordinate system, corrections are made to reduce each measurement of elevation, azimuth, and range to one frame of reference.

It is assumed that during this interval $\bar{r}_0 \cdot \bar{v}_0 = 0$.

$\bar{\rho}_0$ is known and therefore r_0 is known. This allows us to determine the coefficients of \bar{r}_0 and \bar{v}_0 .

With 200 observations, we have 600 equations, 200 equations for each of the components of \bar{r}_0 and \bar{v}_0 .

The form of the equations is:

$$A x_i + B \dot{x}_i = R_{\odot x_i} + \rho_{x_i} \quad i = 1, 2, 3$$

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned}$$

This allows us to solve for \bar{r}_0 and \bar{v}_0 and thus determine the orbit.

II. The Gaussian Method of Orbit Determination.

A. Outline of the Method.

An important difference between the method of Gauss and that of Laplace (section I.A) has to do with what force function one assumes. In the method of Laplace, where substitution for

$$\frac{d^2 \bar{r}}{dt^2}$$

is needed, one could substitute $-\frac{\bar{r}}{r^3} + \bar{F}$,

where \bar{F} is any function called for by the problem. The Gaussian method assumes elliptic motion, $\bar{F} \equiv \bar{0}$.

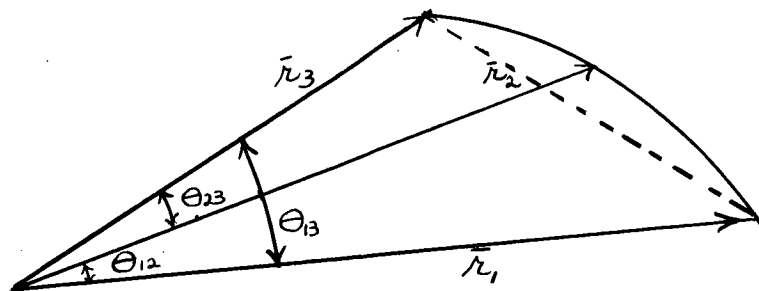


Figure 3.

Since elliptic motion takes place in a plane, we can say

$$(12) \quad \bar{r}_2 = c_1 \bar{r}_1 + c_3 \bar{r}_3$$

If $c_1 + c_3 = 1.0$, the position of the object at \bar{r}_2 would be on the chord joining \bar{r}_1 to \bar{r}_3 . Since the path is curved, $c_1 + c_3 \gg 1.0$.

Using equations (1) and (12) we can write

$$(13) \quad c_1 \rho_1 \hat{\rho}_1 - \rho_2 \hat{\rho}_2 + c_3 \rho_3 \hat{\rho}_3 = c_1 \bar{R}_1 - \bar{R}_2 + c_3 \bar{R}_3$$

If we dot equation (13) first by $(\hat{\rho}_2 \times \hat{\rho}_3)$ and then by $(\rho_1 \times \hat{\rho}_2)$ we obtain the following two expressions

$$(14) \quad c_1 [\hat{\rho}_1 \cdot \hat{\rho}_2 \times \hat{\rho}_3] \rho_1 = c_1 [\bar{R}_1 \cdot \hat{\rho}_2 \times \hat{\rho}_3] - [\bar{R}_2 \cdot \hat{\rho}_2 \times \hat{\rho}_3] + c_3 [\bar{R}_3 \cdot \hat{\rho}_2 \times \hat{\rho}_3]$$

$$(15) \quad c_3 [\hat{\rho}_3 \cdot \hat{\rho}_1 \times \hat{\rho}_2] \rho_3 = c_1 [\bar{R}_1 \cdot \hat{\rho}_1 \times \hat{\rho}_2] - [\bar{R}_2 \cdot \hat{\rho}_1 \times \hat{\rho}_2] + c_3 [\bar{R}_3 \cdot \hat{\rho}_1 \times \hat{\rho}_2]$$

If $\hat{\rho}_1 \cdot \hat{\rho}_2 \times \hat{\rho}_3 = 0$, which implies $\hat{\rho}_3 \cdot \hat{\rho}_1 \times \hat{\rho}_2 = 0$, then we experience the same indeterminacy found in the Laplacian method.

Assuming the equations are not indeterminate, proceed by operating on equation (12). Cross equation (12) by

\bar{r}_3 and dot by \hat{R} . \hat{R} is the unit vector perpendicular to the plane of the orbit.

This gives

$$(16) \quad c_1 = \frac{\bar{r}_2 \times \bar{r}_3 \cdot \hat{R}}{\bar{r}_1 \times \bar{r}_3 \cdot \hat{R}} = \frac{r_2 r_3 \sin \theta_{23}}{r_1 r_3 \sin \theta_{13}} = \frac{[r_2, r_3]}{[r_1, r_3]}$$

Equation (16) will serve as the definition of $[r_1, r_j]$ which is in effect the area of the triangle between \bar{r}_1 and \bar{r}_j . (See Figure 3.)

Then, if we define (r_1, r_j) as the area of the sector of the ellipse between \bar{r}_1 and \bar{r}_j , we can establish a new quantity η_{1j} .

$$\eta_{1j} = \frac{\text{area of sector}}{\text{area of triangle}} = \frac{(r_1, r_j)}{[r_1, r_j]}$$

From (16), we can therefore write

$$c_1 = \frac{\eta_{13}}{(r_1, r_3)} \frac{(r_2, r_3)}{\eta_{2,3}}$$

From Kepler's Law of Areas for elliptic motion, which states that the radius vector sweeps out equal areas in equal intervals of time, we can also say

$$\frac{(r_2, r_3)}{(r_1, r_3)} = \frac{(t_3 - t_2)}{(t_3 - t_1)}$$

Therefore,

$$(17) \quad c_1 = \frac{(t_3 - t_2)}{(t_3 - t_1)} \frac{\eta_{13}}{\eta_{23}}$$

Similarly

$$(18) \quad c_3 = \frac{(t_2 - t_1)}{(t_3 - t_1)} \frac{\eta_{13}}{\eta_{12}}$$

Clearly, if we can find the η 's, we will know c_1 and c_3 . From equations (14) and (15) we can get ρ_1 and ρ_3 . These give \bar{r}_2 . We can then find \bar{v}_2 from some formula such as equation (8) in section (I.C.) and therefore we have calculated the orbit.

B. The Development of the η 's.

Let us define the quantities f and g by the following.
(Herget, beginning at p. 54)

$$\begin{aligned} 2g &= E_j - E_1 & E \text{ is the eccentric anomaly} \\ 2f &= v_j - v_1 & v \text{ is the true anomaly} \end{aligned}$$

From the formulas for elliptic motion,

$$(19) \quad \begin{aligned} r_1 + r_j &= 2a - ae (\cos E_j + \cos E_1) \\ &= 2a \sin^2 g + 2 \sqrt{r_1 r_j} \cos f \cos g. \end{aligned}$$

$$(20) \quad \frac{k(t_j - t_1)}{a^{3/2}} = 2g - e(\sin E_j - \sin E_1) \quad [\text{Kepler's Equation}]$$

$$= 2g - \sin 2g + \frac{2 \sqrt{r_1 r_j}}{a} \cos f \cos g.$$

Now define the quantities ρ , m , and l by

$$\begin{aligned} \rho^2 &= 2(r_1 r_j + \bar{r}_1 \cdot \bar{r}_j) \\ &= 4 r_1 r_j \cos^2 f \\ &= 2(r_1 r_j + x_1 x_j + y_1 y_j + z_1 z_j) \end{aligned}$$

$$(21) \quad m^2 = \frac{[k(t_j - t_1)]^2}{H^3}$$

$$1 + 2l = \frac{r_1 + r_j}{H}$$

or

$$(22) \quad l = \frac{r_1 + r_j - H}{2H}$$

Consider a quantity

$$x = \sin^2 \left(\frac{1}{2} g \right)$$

From equation (19)

$$a = \frac{H(1+x)}{\sin^2 g}$$

and from (20) and (21)

$$k(t_j - t_1) = (2g - \sin 2g)a^{3/2} + (2\sqrt{r_1 r_j} \cos f \sin g) a^{1/2}$$

This implies

$$\frac{(2g - \sin 2g)}{\sin^3 g} (1+x)^{3/2} + (1+x)^{1/2} = \pm m$$

From this last formula, we have

$$(23) \quad 1+x = \frac{m^2}{\eta^2}$$

and therefore

$$(24) \quad \eta = 1 + \frac{m^2}{\eta^2} X(x)$$

$$\text{where } X(x) = \frac{\alpha g - \sin 2g}{\sin^3 g}$$

Expanding $X(x)$

$$\begin{aligned} X(x) &= \frac{2g - (2g - \frac{8g^3}{6} + \dots)}{(g - \frac{g^3}{6} + \dots)^3} \\ &= \frac{\frac{8g^3}{6} + \dots}{g^3 + \dots} \end{aligned}$$

tells us that the constant term in the expansion of $X(x)$ is $4/3$.

From the definition of x ,

$$\frac{dx}{dg} = \frac{1}{2} \sin g.$$

Differentiating $X(x)$

$$\sin^3 g \frac{dX}{dg} + 3 \sin^2 g \cos g X = 2 - 2\cos 2g = 4 \sin^2 g,$$

we get

$$\frac{dX}{dg} = \frac{4 - 3 \cos g X}{\sin g}$$

If we write the following

$$\frac{dX}{dx} = \frac{dX}{dg} \frac{dg}{dx} = \frac{8 - 6 \cos g X}{\sin^2 g} = \frac{4 - 3(1-2x) X}{2x(1-x)},$$

we obtain

$$(25) \quad (2x - 2x^2) \frac{dX}{dx} = 4 - 3(1 - 2x) X$$

Assume X can be expanded in a series of the form

$$X(x) = \sum_{n=0}^{\infty} A_n x^n$$

Then,

$$\frac{dX}{dx} = \sum_{n=1}^{\infty} n A_n x^{n-1}$$

Substituting these forms into equation (25) and equating coefficients of powers of x^n , we obtain the following recurrence relation.

$$A_n = \frac{2n+4}{2n+3} A_{n-1}$$

$A_0 = 4/3$, from a previous argument.

Therefore,

$$(26) \quad X(x) = \frac{4}{3} + \frac{4}{3} \cdot \frac{6}{5} x + \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} x^2 + \dots$$

Recall that $\sin^2(\frac{1}{2}g) = \frac{1}{2}$, where $2g = E_j - E_1$.

Therefore, $\frac{1}{2}g = \frac{1}{4} \Delta E$. If ΔE is small, x is also small and the convergence of $X(x)$ is rapid.

The equations (23) and (24) can be solved for η in an iterative manner. A possible starting point is to set $\eta = 1.0$. Then find m^2 and \mathbf{l} from (21) and (22). Equation (23) gives x which in turn gives a new value of η from (24).

The following is an outline of a FORTRAN program to compute $X(x)$.

Initialize $X = 4.0/3.0$

P.T. = $4.0/3.0$

EN = 4.0

D = 3.0

P.T. = previous term

EN = numerator

D = demoninator

→ EN = EN + 2.0

D = D + 2.0

P.T. = P.T. * (EN/D) * x

X = X + P.T.

Test P.T. against a tolerance.

When X is carried as far as necessary, one can go on to compute a new value of η . This continues until convergence of equations (23) and (24), using (21) and (22).

C. Determining the Orbit

Assume now that n observations have been made. The method followed here is a new version of the variation of geocentric distances. ρ_1 and ρ_n are guessed and then allowed to change.

Look at equation (13), modified to let $\rho_1 \hat{\rho}_1$ stand for any of the middle observations.

$$c_1 \rho_1 \hat{\rho}_1 - \rho_1 \hat{\rho}_1 + c_n \rho_n \hat{\rho}_n = c_1 \bar{R}_1 - \bar{R}_1 + c_n \bar{R}_n$$

We wish to consider the effect of small changes $\Delta\rho_1$, and $\Delta\rho_n$ on this equation. Including these effects, and dotting the equation by \hat{A}_1 , we obtain

$$\begin{aligned} & \left(c_1 + \frac{\partial c_1}{\partial \rho_1} \Delta\rho_1 + \frac{\partial c_1}{\partial \rho_n} \Delta\rho_n \right) (\rho_1 + \Delta\rho_1) \left[\hat{\rho}_1 \cdot \hat{A}_1 \right] \\ & + \left(c_n + \frac{\partial c_n}{\partial \rho_1} \Delta\rho_1 + \frac{\partial c_n}{\partial \rho_n} \Delta\rho_n \right) (\rho_n + \Delta\rho_n) \left[\hat{\rho}_n \cdot \hat{A}_1 \right] \\ & = \left(c_1 + \frac{\partial c_1}{\partial \rho_1} \Delta\rho_1 + \frac{\partial c_1}{\partial \rho_n} \Delta\rho_n \right) \left[\bar{R}_1 \cdot \hat{A}_1 \right] \\ & + \left(c_n + \frac{\partial c_n}{\partial \rho_1} \Delta\rho_1 + \frac{\partial c_n}{\partial \rho_n} \Delta\rho_n \right) \left[\bar{R}_n \cdot \hat{A}_1 \right] - \left[\bar{R}_1 \cdot \hat{A}_1 \right] \end{aligned}$$

Collecting terms:

$$\begin{aligned} (27) \quad & \left\{ \left(c_1 + \frac{\partial c_1}{\partial \rho_1} \rho_1 \right) \left[\hat{\rho}_1 \cdot \hat{A}_1 \right] + \frac{\partial c_n}{\partial \rho_1} \rho_n \left[\hat{\rho}_n \cdot \hat{A}_1 \right] \right. \\ & \left. - \frac{\partial c_1}{\partial \rho_1} \left[\bar{R}_1 \cdot \hat{A}_1 \right] - \frac{\partial c_n}{\partial \rho_1} \left[\bar{R}_n \cdot \hat{A}_1 \right] \right\} \Delta\rho_1 \\ & + \left\{ \left(c_n + \frac{\partial c_n}{\partial \rho_n} \rho_n \right) \left[\hat{\rho}_n \cdot \hat{A}_1 \right] + \frac{\partial c_1}{\partial \rho_n} \rho_1 \left[\hat{\rho}_1 \cdot \hat{A}_1 \right] \right. \\ & \left. - \frac{\partial c_1}{\partial \rho_n} \left[\bar{R}_1 \cdot \hat{A}_1 \right] - \frac{\partial c_n}{\partial \rho_n} \left[\bar{R}_n \cdot \hat{A}_1 \right] \right\} \Delta\rho_n \\ & = c_1 \left[\bar{R}_1 \cdot \hat{A}_1 \right] - \left[\bar{R}_1 \cdot \hat{A}_1 \right] + c_n \left[\bar{R}_n \cdot \hat{A}_1 \right] \\ & - c_1 \rho_1 \left[\hat{\rho}_1 \cdot \hat{A}_1 \right] - c_n \rho_n \left[\hat{\rho}_n \cdot \hat{A}_1 \right] \end{aligned}$$

Terms of order Δ^2 are neglected.

A second equation is obtained by replacing \hat{A}_1 by \hat{D}_1 in (27).

The method is described for use on a computer. List each observation with its corresponding Julian date and \bar{R} vector. (\bar{R} may need to be corrected for topocentric parallax). Read in the last observation first and the first observation second. Then guess a value for ρ_1 and ρ_n . Also guess an initial value of Δ .

With this ρ_1 and ρ_n , the machine computes η_{1n} , c_1 and c_n .

The remaining $\hat{\rho}_1$ are now read in one at a time, with all of the bracketed terms in equation (27) being evaluated each time. The partials of c are evaluated by considering

$\rho_1 \pm \Delta$ and $\rho_n \pm \Delta$. $\rho_1 \pm \Delta$ yields two slightly different values of c_1 and c_n from those given by ρ_1 . Call these values c_{1+} , c_{1-} , c_{n+} , and c_{n-} .

Then

$$\frac{c_{1+} - c_{1-}}{2\Delta} = \frac{\partial c_1}{\partial \rho_1}$$

$$\frac{c_{n+} - c_{n-}}{2\Delta} = \frac{\partial c_n}{\partial \rho_1}$$

Similarly, $\rho_n \pm \Delta$ will give different values of c_1 and c_n .

$$\frac{c_{1+} - c_{1-}}{2\Delta} = \frac{\partial c_1}{\partial \rho_n}$$

$$\frac{c_{n+} - c_{n-}}{2\Delta} = \frac{\partial c_n}{\partial \rho_n}$$

With these partials we have two equations of the form

$$\alpha_{A_i} \Delta \rho_1 + \beta_{A_i} \Delta \rho_n = \gamma_{A_i}$$

$$\alpha_{D_i} \Delta \rho_1 + \beta_{D_i} \Delta \rho_n = \gamma_{D_i}$$

in two unknowns $\Delta \rho_1$ and $\Delta \rho_n$. If there are n observations there are $n - 2$ pairs of such equations. A least squares solution will give results provided

$$\begin{vmatrix} \alpha_A & \beta_A \\ \alpha_D & \beta_D \end{vmatrix} \neq 0$$

If this value of the determinant is read out along with the values of $\Delta \rho_1$, and $\Delta \rho_n$, then the value of the solution can be judged. If they are good, then continue to compute the final orbit with these final values of $\bar{\rho}_1$ and $\bar{\rho}_n$. If they are not good, continue the above process.

In this solution, you are assuming that the guessed values of ρ_1 and ρ_n are giving you the correct orbit. This orbit will determine for you a $\hat{\rho}_1$ based only on the values of $\bar{\rho}_1$ and $\bar{\rho}_n$. The observed $\hat{\rho}_1$, used in the form of \hat{A}_1 and \hat{D}_1 measure the error you are making in assuming ρ_1 and ρ_n are correct.

Once you are satisfied with a $\bar{\rho}_1$ and $\bar{\rho}_n$, you can compute \bar{r}_n , \bar{r}_1 , \bar{r}_o (where $t = t_o$ is some epoch you have chosen).

The following equations can be used to compute \bar{v}_o .

$$\bar{r}_n = f_n \bar{r}_o + g_n \bar{v}_o$$

$$\bar{r}_1 = f_1 \bar{r}_o + g_1 \bar{v}_o$$

Therefore

$$\bar{v}_o = \frac{\bar{r}_n - \bar{r}_1 + (f_1 - f_n) \bar{r}_o}{g_n - g_1}$$

$$\text{where } g_1 = \frac{k(t_1 - t_o)}{\eta_{10}} \quad ; \quad g_n = \frac{k(t_n - t_o)}{\eta_{on}}$$

$$f_1 = \frac{1 - 2 [k(t_o - t_1)]^2}{r_o \eta_{10}^2 H_{10}^2}$$

$$f_n = \frac{1 - 2 [k(t_n - t_o)]^2}{r_o \eta_{on}^2 H_{on}^2}$$

(Herget, p. 57 bottom).

Look carefully at the residuals from the solutions for $\Delta \rho_1$ and $\Delta \rho_n$. Those from the first and last observations will be zero. If a pattern appears in the remaining residuals, it may be possible to juggle the figures to get a better fit.

III. Improvement of the Orbit (without perturbations).

There are many methods which can be used to improve the preliminary orbit, which is found by methods discussed in Sections I and II. After a large number of observations have accumulated, one is in a position to apply corrections. The orbit is a conic section and one can apply corrections either to its elements (a , e , i , etc.) or to the initial position and velocity vectors at some time, t_0 . It is also useful to have formulae which correct an orbit which is not a conic section, but is perturbed. These formulae will be discussed in Section IV.

A. Corrections from a Small Change in Initial Conditions.

The technique to be used is that of undetermined differential variations ($\delta \bar{r}_0$ and $\delta \bar{v}_0$) of the initial conditions (\bar{r}_0 and \bar{v}_0). Then at t_0 we have

$$\bar{r}_{01} = \bar{r}_0 + \delta \bar{r}_0$$

$$\bar{v}_{01} = \bar{v}_0 + \delta \bar{v}_0 \quad , \quad \text{where } \delta \bar{r}_0 \text{ and } \delta \bar{v}_0 \text{ are as yet unknown.}$$

$$\text{From equation (8), } \bar{r} = f \bar{r}_0 + g \bar{v}_0,$$

we can write the first variational equation,

$$(28) \quad \delta \bar{r} = f \delta \bar{r}_0 + g \delta \bar{v}_0 + \delta f \bar{r}_0 + \delta g \bar{v}_0$$

This can also be expressed in matrix form,

$$(29) \quad \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \delta \bar{r} = \begin{bmatrix} \frac{\partial x}{\partial x_0} \frac{\partial x}{\partial y_0} & . & . & . & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & . & . & . & . \\ \frac{\partial z}{\partial x_0} & . & . & . & \frac{\partial z}{\partial z_0} \end{bmatrix} \begin{bmatrix} \delta x_0 \\ \delta y_0 \\ \delta z_0 \\ \delta x'_0 \\ \delta y'_0 \\ \delta z'_0 \end{bmatrix}$$

The notation to be used is $\bar{v}_0 = (x'_0, y'_0, z'_0)$.

The plan is to express f and g in terms of variables in which they are easily differentiated and then change variables to those ultimately required by the problem.

We shall use the following expressions in the development.

$$\frac{1}{2} \delta(\bar{r}_0 \cdot \bar{r}_0) = \bar{r}_0 \cdot \delta \bar{r}_0 = r_0 \delta r_0 = x_0 \delta x_0 + y_0 \delta y_0 + z_0 \delta z_0$$

$$\begin{aligned} \frac{1}{2} \delta(\bar{v}_0 \cdot \bar{v}_0) &= \frac{1}{2} \delta(v_0^2) = x'_0 \delta x'_0 + y'_0 \delta y'_0 \\ &+ z'_0 \delta z'_0 \end{aligned}$$

$$\begin{aligned} \delta(\bar{r}_0 \cdot \bar{v}_0) &= \delta(D_0) = x_0 \delta x'_0 + y_0 \delta y'_0 + z_0 \delta z'_0 \\ &+ x'_0 \delta x_0 + y'_0 \delta y_0 + z'_0 \delta z_0 \end{aligned}$$

Let $\Delta E = E - E_0$.

Then, let

$$F = a(1 - \cos \Delta E)$$

and

$$G = \sqrt{a} \sin \Delta E$$

Therefore

$$\delta F = \frac{F}{a} \delta a + \sqrt{a} G \delta \Delta E$$

$$\delta G = \frac{1}{2a} G \delta a + \sqrt{a} \cos \Delta E \delta \Delta E.$$

The quantity f , may be written

$$f = 1 - \frac{F}{r_0}$$

$$\delta f = \frac{F}{r_0^2} \delta r_0 - \frac{F}{ar_0} \delta a - \frac{\sqrt{a}}{r_0} G \delta \Delta E$$

To find δa , consider the energy integral (the unit of time being chosen such that $\mu = 1.0$).

$$\frac{1}{a} = \frac{2}{r_0} - v_0^2$$

$$\frac{\delta a}{a} = \frac{2 \delta r_0}{r_0^2} + \delta(v_0^2)$$

To eliminate $\delta \Delta E$, we use two separate expressions for g .

$$g = k(t - t_0) - a^{3/2} [\Delta E - \sin \Delta E]$$

$$\delta g = -\frac{3}{2} \left[\frac{k(t - t_0) - g(\delta a)}{a} \right] - \sqrt{a} F \delta \Delta E$$

$$g = G r_0 + F D_0$$

$$\delta g = G \delta r_0 + F \delta D_0 + \left[\frac{r_0 G}{2} + D_0 F \right] \frac{\delta a}{a}$$

$$+ \left[r_0 \sqrt{a} \cos \Delta E + D_0 \sqrt{a} G \right] \delta \Delta E$$

Therefore,

$$-\sqrt{a} \delta \Delta E = \frac{G}{r} \delta r_0 + \frac{aL}{2} \frac{\delta a}{a^2} + \frac{F}{r} \delta D_0,$$

where

$$L = \frac{3k(t-t_0) - g - G r_0}{r}$$

The previous development is based on the presentation of Bower.

We now have expressions for δf and δg in terms of \bar{r}_0 , \bar{v}_0 , r and $\delta \bar{r}_0$, $\delta \bar{v}_0$. The complete expressions are found in Herget, pp. 74 ff. (See equations ((6, 6)) and ((6, 10)) and recall that these are written in Cracovians).

The equation to be solved for the unknowns $\delta \bar{r}_0$, $\delta \bar{v}_0$ is ((6, 8)) (Herget, p. 75). It shall be derived now. In what follows, all quantities are assumed to be in the equatorial system.

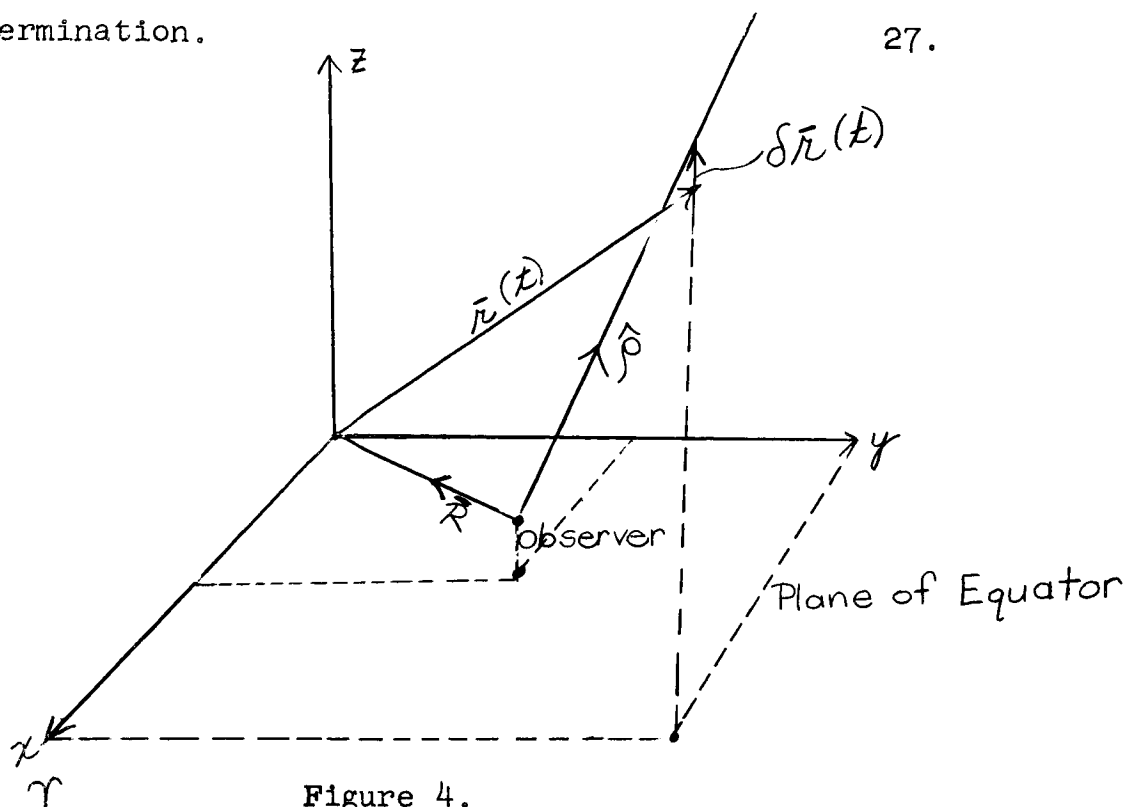


Figure 4.

From, $\bar{r} = \bar{\rho} - \bar{R}$, we get

$$(30) \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \delta \bar{r} = \delta \bar{\rho} = \begin{bmatrix} \uparrow \\ \hat{A} \\ \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \hat{D} \\ \downarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \hat{\rho} \\ \downarrow \end{bmatrix} \begin{bmatrix} \rho \cos \delta \Delta \alpha \\ \rho \Delta \delta \\ \Delta \rho \end{bmatrix}$$

(Herget, p. 73, Equation ((6,3))).

$\Delta \alpha$, $\Delta \delta$, and $\Delta \rho$ are observed minus computed (O - c) residuals.

If we pre-multiply equation (30) first by \hat{A} , then \hat{D} and then $\hat{\rho}$, we obtain the following.

$$(31) \begin{aligned} \hat{A} \cdot \delta \bar{r} &= \rho \cos \delta \Delta \alpha & (O - c) \\ \hat{D} \cdot \delta \bar{r} &= \rho \Delta \delta & (O - c) \\ \hat{\rho} \cdot \delta \bar{r} &= \Delta \rho & (O - c) \end{aligned}$$

Such residuals exist for each observation. We wish to correct the orbit in such a way that $\Delta(0 - c)$ are made as small, nearly zero, as possible.

From (30) $\delta \bar{r} = \delta \bar{\rho} = \bar{\rho} \text{ (observed)} - \bar{\rho} \text{ (computed)}$.
By definition, it is true that

$$\hat{A} \cdot \hat{\rho} \text{ (observed)} = 0 = \hat{D} \cdot \hat{\rho} \text{ (observed)}.$$

$$\begin{aligned} \text{Therefore } \hat{A} \cdot \delta \bar{r} &= \hat{A} \cdot \delta \bar{\rho} = \hat{A} \cdot [\bar{\rho} \text{ (observed)} - \bar{\rho} \text{ (computed)}] \\ &= - \hat{A} \cdot \bar{\rho} \text{ (computed)} \\ &= - \hat{A} \cdot [\bar{r} \text{ (computed)} + \bar{R}] \end{aligned}$$

We can therefore rewrite (31) as

$$\begin{aligned} (32) \quad - \frac{\hat{A} \cdot (\bar{r} + \bar{R})}{\rho} &= \cos \delta \Delta \alpha \quad (0 - c) \\ - \frac{\hat{D} \cdot (\bar{r} + \bar{R})}{\rho} &= \Delta \delta \quad (0 - c) \end{aligned}$$

The third relation is not used since in most cases ρ is not observed directly. Equations (32) allow us to compute the residuals for each observation. To tie these results to our expressions for $\delta \bar{r}_0$ and $\delta \bar{v}_0$, recall that (31) can be written in matrix form:

$$(33) \quad \frac{1}{\rho} \begin{bmatrix} \leftarrow \hat{A} \rightarrow \\ \leftarrow \hat{D} \rightarrow \end{bmatrix} \begin{bmatrix} \delta \bar{r} \end{bmatrix} = \begin{pmatrix} \cos \delta \Delta \alpha \\ \Delta \delta \end{pmatrix}$$

Substituting for $\delta \bar{r}$, from (29) we obtain one method for computing the variations.

$$(34) \quad \frac{1}{\rho} \begin{bmatrix} \leftarrow \hat{A} \rightarrow \\ \leftarrow \hat{D} \rightarrow \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial x_0} & \dots & \frac{\partial x}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \dots & \frac{\partial z}{\partial z_0} \end{bmatrix} \begin{bmatrix} \delta x_0 \\ \delta y_0 \\ \vdots \\ \delta z_0 \end{bmatrix} = \begin{pmatrix} \cos \delta \Delta \alpha \\ \Delta \delta \end{pmatrix}$$

This is equation ((6,8)) as it appears in Herget, p. 75.

(In the book it is written in Cracovians.)

B. Method of Solving System (34).

With several observations available, we have enough information to perform a least squares solution to system (34).

Each equation will be of the form

$$(35) \quad a x_1 + b x_2 + c x_3 + d x_4 + e x_5 + f x_6 = m$$

where $x_1 = \delta x_0, \dots, x_6 = \delta z_0$.

The following elimination method is suggested for computational work. The solutions and probable error are found in one process.

From (35), set up the normal equations

$$\begin{bmatrix} aa \\ ba \\ \vdots \\ fa \end{bmatrix} x_1 + \begin{bmatrix} ab \\ bb \\ \vdots \\ fb \end{bmatrix} x_2 + \dots + \begin{bmatrix} af \\ \vdots \\ ff \end{bmatrix} x_6 = \begin{bmatrix} am \\ bm \\ \vdots \\ fm \end{bmatrix}$$

When the computation is complete, we would have two matrices A and B which appear as follows.

$\begin{bmatrix} aa & ab & ac & ad & ae & af \\ ba & bb & . & . & . & . \\ ca & . & cc & . & . & . \\ da & . & . & dd & . & . \\ ea & . & . & . & ee & . \\ fa & . & . & . & . & ff \end{bmatrix} : \begin{bmatrix} am \\ bml \\ . \\ . \\ . \\ fm \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & . & . & . & . \\ 0 & . & 1 & . & . & . \\ 0 & . & . & 1 & . & . \\ 0 & . & . & . & 1 & . \\ 0 & . & . & . & . & 1 \end{bmatrix}$
$\begin{bmatrix} aa & ab & ac & ad & ae & af \\ & bbl & bcl & bdl & bel & bfl \\ & & cc2 & cd2 & ce2 & cf2 \\ & & & dd3 & de3 & df3 \\ & & & & ee4 & ef4 \\ & & & & & ff5 \end{bmatrix} : \begin{bmatrix} am \\ bml \\ . \\ . \\ . \\ fm \end{bmatrix}$	$\begin{matrix} A_1 & B_1 & C_1 & D_1 & E_1 \\ & B_2 & C_2 & D_2 & E_2 \\ & & C_3 & D_3 & E_3 \\ & & & D_4 & E_4 \\ & & & & E_5 \end{matrix}$

(A)
(B)

On the computer one starts with the upper half of A (the normal equations) and with the upper half of B (the identity matrix). If we multiply every column of A by column 1 of B we obtain Row 7 of A. Then, dividing every member of row 7 in A by $[aa]$, we obtain row 7 in B.

$$\text{For example, } A_1 = \frac{[ab]}{[aa]}, \quad B_1 = \frac{[ac]}{[aa]}, \text{ etc.}$$

We now multiply every column in A by column 2 of B. This gives row 8 of A.

$$\begin{aligned} \text{For example, } \begin{bmatrix} bbl \\ bcl \end{bmatrix} &= \begin{bmatrix} bb \\ bc \end{bmatrix} \cdot 1 + \begin{bmatrix} ab \\ ac \end{bmatrix} \cdot A_1 + 0 \\ &\text{etc.} \end{aligned}$$

Dividing row 8 of A by $[bb1]$ we obtain row 8 of B.

$$B_2 = \frac{\begin{bmatrix} bcl \\ bbl \end{bmatrix}}{\begin{bmatrix} bcl \\ bbl \end{bmatrix}}, \text{ etc.}$$

This process continues until the whole of matrices A and B are computed.

The lower half of A gives the solutions for X_1, \dots, x_6 , starting with

$$x_6 = \frac{\begin{bmatrix} fm5 \\ ff5 \end{bmatrix}}{\begin{bmatrix} fm5 \\ ff5 \end{bmatrix}}$$

We then compute the quantities,

$$\begin{aligned} A_2 &= B_1 - A_1 B_2 \\ A_3 &= C_1 - A_1 C_2 - A_2 C_3 \\ A_4 &= D_1 - A_1 D_2 - A_2 D_3 - A_3 D_4 \\ A_5 &= E_1 - A_1 E_2 - A_2 E_3 - A_3 E_4 - A_4 E_5 \end{aligned}$$

$$\begin{aligned} B_3 &= C_2 - B_2 C_3 \\ B_4 &= D_2 - B_2 D_3 - B_3 D_4 \\ B_5 &= E_2 - B_2 E_3 - B_3 E_4 - B_4 E_5 \end{aligned}$$

$$\begin{aligned} C_4 &= D_3 - C_3 D_4 \\ C_5 &= E_3 - C_3 E_4 - C_4 E_5 \end{aligned}$$

$$D_5 = E_4 - D_4 E_5$$

From these we obtain,

$$\begin{aligned}
 Q_{66} &= \frac{1}{[ff5]} \\
 Q_{55} &= \frac{1}{[ee4]} + \frac{E_5^2}{[ff5]} \\
 Q_{44} &= \frac{1}{[dd3]} + \frac{D_4^2}{[ee4]} + \frac{D_5^2}{[ff5]} \\
 Q_{33} &= \frac{1}{[cc2]} + \frac{C_3^2}{[dd3]} + \frac{C_4^2}{[ee4]} + \frac{C_5^2}{[ff5]} \\
 Q_{22} &= \frac{1}{[bb1]} + \frac{B_2^2}{[cc2]} + \frac{B_3^2}{[dd3]} + \frac{B_4^2}{[ee4]} + \frac{B_5^2}{[ff5]} \\
 Q_{11} &= \frac{1}{[aa]} + \frac{A_1^2}{[bb1]} + \frac{A_2^2}{[cc2]} + \frac{A_3^2}{[dd3]} + \frac{A_4^2}{[ee4]} + \frac{A_5^2}{[ff5]}
 \end{aligned}$$

The probable errors of unit weight for each measurement are given by

$$(36) \quad p.e. \ x_i = 0.6745 \sqrt{\frac{[vv] \ Q_{11}}{n - k}}$$

$[vv]$ = the sum of the squares of the residuals.
 The residuals are obtained by substituting the final values of x_i into the equations of condition (35).
 Each residual is therefore found by

$$m_j - \sum_{i=1}^6 \alpha_i x_i \quad (\alpha_i = \text{coefficient of } x_i).$$

$$\text{and } [vv] = \sum_{j=1}^n (m_j - \sum_{i=1}^6 \alpha_i x_i)^2 \quad \text{where the } \alpha_i \text{ may be different for each } j.$$

n = the number of equations of condition available.

k = the number of unknowns, in this case, $k = 6$.

If $n = 6$, the probable error will show it. This is the case where 3 observations are used in System (33) to correct an orbit.

If $Q_{ii} \approx 1$, then the observations are favorable. If there is a large correlation between unknowns through the coefficients, Q_{ii} will be larger. This is the case when the system is physically ill-conditioned.

One more note on the computation is made in the consideration of computer space.

The lower halves of matrices A and B can be condensed in the following manner.

$$\begin{array}{ccccccc}
 \begin{bmatrix} aa \\ A_1 \end{bmatrix} & \begin{bmatrix} ab \\ B_1, A_2 \end{bmatrix} & \begin{bmatrix} ac \\ C_1, A_3 \end{bmatrix} & \begin{bmatrix} ad \\ D_1, A_4 \end{bmatrix} & \begin{bmatrix} ae \\ E_1, A_5 \end{bmatrix} & \begin{bmatrix} af \\ F_1, A_6 \end{bmatrix} & : \begin{bmatrix} am \\ M_1, A_7 \end{bmatrix} \\
 & \begin{bmatrix} bbl \\ B_2 \end{bmatrix} & \begin{bmatrix} bcl \\ C_2, B_3 \end{bmatrix} & \begin{bmatrix} bdl \\ D_2, B_4 \end{bmatrix} & \begin{bmatrix} bel \\ E_2, B_5 \end{bmatrix} & \begin{bmatrix} bfl \\ F_2, B_6 \end{bmatrix} & : \begin{bmatrix} bml \\ M_2, B_7 \end{bmatrix} \\
 & \begin{bmatrix} ccl \\ C_3 \end{bmatrix} & \begin{bmatrix} cdl \\ D_3, C_4 \end{bmatrix} & \begin{bmatrix} cel \\ E_3, C_5 \end{bmatrix} & \begin{bmatrix} cfl \\ F_3, C_6 \end{bmatrix} & \begin{bmatrix} cml \\ M_3, C_7 \end{bmatrix} & : \begin{bmatrix} cml \\ M_3, C_7 \end{bmatrix} \\
 & \begin{bmatrix} dcl \\ D_4 \end{bmatrix} & \begin{bmatrix} del \\ E_4, D_5 \end{bmatrix} & \begin{bmatrix} dfl \\ F_4, D_6 \end{bmatrix} & \begin{bmatrix} dml \\ M_4, D_7 \end{bmatrix} & \begin{bmatrix} dml \\ M_4, D_7 \end{bmatrix} & : \begin{bmatrix} dml \\ M_4, D_7 \end{bmatrix} \\
 & \begin{bmatrix} eel \\ E_5 \end{bmatrix} & \begin{bmatrix} efl \\ F_5, E_6 \end{bmatrix} & \begin{bmatrix} eml \\ M_5, E_7 \end{bmatrix} & \begin{bmatrix} eml \\ M_5, E_7 \end{bmatrix} & \begin{bmatrix} eml \\ M_5, E_7 \end{bmatrix} & : \begin{bmatrix} eml \\ M_5, E_7 \end{bmatrix} \\
 & \begin{bmatrix} ffl \\ F_6 \end{bmatrix} & \begin{bmatrix} fml \\ M_6, F_7 \end{bmatrix} & \begin{bmatrix} fml \\ M_6, F_7 \end{bmatrix} & \begin{bmatrix} fml \\ M_6, F_7 \end{bmatrix} & \begin{bmatrix} fml \\ M_6, F_7 \end{bmatrix} & : \begin{bmatrix} fml \\ M_6, F_7 \end{bmatrix}
 \end{array}
 ,$$

where, for example, A_2 can replace B_1 as soon as it is computed. The identity matrix need not be stored at all.

C. Corrections to Orbits Determined by Radar

(This outline illustrates the principle involved when the observations are not weighted equally.)

Since we have established equations (29) with respect to an equatorial system, we need to refer to another

system, that of the observer. In this system z is directed toward the local zenith, x toward the East point on the horizon, and y toward the North point.

The equation corresponding to (33) becomes

$$(37) \quad \frac{1}{\rho} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \Lambda \end{bmatrix} \begin{bmatrix} \delta \vec{r} \end{bmatrix} = \begin{pmatrix} \cos H \Delta \rho \\ \Delta H \end{pmatrix} \begin{bmatrix} A \end{bmatrix} \quad (0 - c)$$

where

$$A = \begin{pmatrix} \rho \cos H \sin A & \rho \cos H \cos A & \rho \sin H \\ \cos A & -\sin A & 0 \\ -\sin H \sin A & -\sin H \cos A & \cos H \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} -\sin \lambda & \cos \lambda & 0 \\ -\sin \phi \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \\ \cos \phi \cos \lambda & \cos \phi \sin \lambda & \sin \phi \end{pmatrix}$$

Λ represents a rotation matrix taking us from the system of the equator to that of the observer. ϕ is the astronomical latitude and λ , the astronomical longitude, of the observer.

Note that in this set of equations, $\Delta \rho$ is included among the residuals, since ρ is a basic measured quantity along with A , the azimuth angle and H , the elevation.

The solution will proceed as before. The epoch of time is usually taken to be that time at which the satellite is at a minimum distance from the station.

Involved in the solution is a quantity called the weighting factor. Radar equipment can measure distances better than angles, especially when the object is near the zenith.

Suppose the r.m.s. error, ϵ' , of distance measurement to be about 100 feet. Scaled to the radius of the earth, u , we obtain

$$\frac{\epsilon'}{u} = \epsilon \approx \frac{100 \text{ ft.}}{4000 \text{ mi.}}$$

Let ϕ be the error of the angular measurements. The linear effect is then $\rho\phi$. We are obliged to introduce a weighting factor into the equations

$$\sqrt{w_t} = \sqrt{\frac{\epsilon^2}{(\rho\phi)^2}}$$

If all equations in (37) involving angular measurements are multiplied by this value, then we have given unit weight to the distance measurements and we can expect the errors in each component of the equations to be approximately equal. As A and H are given less and less weight, their probable errors decrease.

IV. Orbit Correction with Perturbations

A. Form of the Equations

In this section, we consider deviations from motion in a conic section. Many of the previous methods can be applied, where the equations of two body motion

are replaced by

$$\frac{d^2 \bar{r}}{dt^2} = -\frac{\bar{r}}{r^3} + \bar{R} = \bar{F}(\bar{r}, \dot{\bar{r}}, c, t) .$$

In planetary motions, $\dot{\bar{r}}$ is absent from \bar{F} , but in artificial satellite motion with drag (for example), $\dot{\bar{r}}$ is present. The quantity C is a constant parameter of the problem, a small quantity.

Let $x_i = x, y, z$, where $i = 1, 2, 3$.

Then, consider the partials of \ddot{x}_i with respect to any quantity, q , which is not a function of time. Then,

$$\frac{\partial \ddot{x}_i}{\partial q} = \sum_j \left(\frac{\partial F_i}{\partial x_j} \frac{\partial x_j}{\partial q} + \frac{\partial F_i}{\partial \dot{x}_j} \frac{\partial \dot{x}_j}{\partial q} + \frac{\partial F_i}{\partial C} \frac{\partial C}{\partial q} \right)$$

$$\frac{\partial q}{\partial t} \equiv 0$$

The variation of the F_i , $i = 1, 2, 3$, with respect to the quantity C is included for purposes of determining an improved value of C in correction. Then, we associate with q the quantities

$$X_j = x_0, y_0, z_0 \quad j = 1, 2, 3$$

$$V_k = x_0', y_0', z_0' \quad k = 1, 2, 3$$

the initial conditions.

Define

$$w_{1j} = \frac{\partial x_1}{\partial x_j} \quad ; \quad \dot{w}_{1j} = \frac{d}{dt} \left(\frac{\partial x_1}{\partial x_j} \right)$$

and

$$w_{1k} = \frac{\partial x_1}{\partial v_k} \quad ; \quad \dot{w}_{1k} = \frac{d}{dt} \left(\frac{\partial x_1}{\partial v_k} \right)$$

These w 's express the partial differential coefficients of the variation of \bar{r} with the initial conditions. At this time one could note the similarity of this development with that in Section III,A. Now equation (28) has added to it a quantity $\oint(\oint \bar{R} dt dt)$.

From the w 's, we can write

$$\begin{aligned} \ddot{w}_{1j} &= \frac{d^2}{dt^2} \left(\frac{\partial x_1}{\partial x_j} \right) = \frac{\partial \ddot{x}_1}{\partial x_j} \\ (38) \quad &= \sum_n \left(\frac{\partial F_1}{\partial x_n} w_{nj} + \frac{\partial F_1}{\partial \dot{x}_n} \dot{w}_{1j} \right) \\ \dot{w}_{1k} &= \frac{\partial \ddot{x}_1}{\partial v_k} \\ &= \sum_n \left(\frac{\partial F_1}{\partial x_n} w_{nk} + \frac{\partial F_1}{\partial \dot{x}_n} \dot{w}_{nk} \right) \end{aligned}$$

The order of differentiation can be changed in (38) since x_j is independent of time. c does not appear in (38) since x_j is independent of c . We can, however,

establish the following equation containing c .

$$(39) \quad \ddot{w}_{ic} = \frac{d^2}{dt^2} \frac{\partial x_1}{\partial c} = \frac{\partial \ddot{x}_1}{\partial c}$$

$$= \sum_n \left(\frac{\partial F_1}{\partial x_n} w_{nc} + \frac{\partial F_1}{\partial \dot{x}_n} \dot{w}_{nc} \right) + \frac{\partial F}{\partial c}$$

Equations (38) and (39) can be numerically integrated to give w_{ij} , w_{ik} , and w_{ic} . The initial conditions are:

$$\text{for } \underline{w_{ij}} : \quad w_{11} = 1.0, \quad w_{ij} = 0.0 \text{ for } i \neq j$$

$$w_{22} = 1.0$$

$$w_{33} = 1.0 \quad \dot{w}_{ij} = 0.0$$

$$\text{for } \underline{w_{ik}} : \quad \dot{w}_{11} = 1.0 \quad w_{ik} = 0.0$$

$$\dot{w}_{22} = 1.0$$

$$\dot{w}_{33} = 1.0 \quad \dot{w}_{ik} = 0.0 \text{ for } i \neq k$$

$$\text{for } \underline{w_{ic}} : \quad w_{ic} = 0.0$$

$$\dot{w}_{ic} = 0.0$$

We have now established a method for computing $\delta \bar{r}$, in a form similar to (29). We have

$$(40) \quad \delta \bar{r} = \begin{bmatrix} \cdot & \cdot & \frac{\partial x_1}{\partial c} \\ w_{1j} & \cdot & \frac{\partial x_2}{\partial c} \\ (3 \times 3) & w_{1k} & \frac{\partial x_3}{\partial c} \\ \cdot & (3 \times 3) & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \delta x \\ \delta v \\ \delta c \end{bmatrix}$$

After one has computed a perturbed orbit, one can put the expression (40) for \bar{r} into equation (33) and apply corrections to a perturbed orbit. If one does not wish to include a correction for c , this part is easily eliminated from the equation.

B. The Satellites of Jupiter

The equations of motion for a satellite, with Jupiter as origin, can be written as

$$(41) \quad \frac{d^2 \bar{r}}{dt^2} = -m_4 k^2 \frac{\bar{r}}{r^3} + M_{\odot} k^2 \left[\frac{\bar{r}_{\odot} - \bar{r}}{\Delta_{\odot}^3} - \frac{\bar{r}_{\odot}}{r_{\odot}^3} \right] \\ + m_h k^2 \left[\frac{\bar{r}_h - \bar{r}}{\Delta_h^3} - \frac{\bar{r}_h}{r_h^3} \right]$$

m_4 : mass of Jupiter

m_h : mass of Saturn

M_{\odot} : mass of Sun

\bar{r} : radius vector from Jupiter to the Satellite

\bar{r}_{\odot} : radius vector from Jupiter to the Sun

\bar{r}_h : radius vector from Jupiter to Saturn

Δ_{\odot} : distance from Sun to Satellite

Δ_h : distance from Saturn to Satellite

To determine the orbit, we first make use of this device of Encke. The bracketed terms in (41) are differences

of very nearly equal quantities. To avoid this problem,

Define q by the relation

$$\frac{\Delta_{\odot}^2}{r_{\odot}^2} = 1 - 2q_{\odot} = 1 - 2 \frac{\bar{r}_{\odot} \cdot \bar{r} - r^2/2}{r_{\odot}^2}$$

Define, f , by

$$fq = 1 - (1 - 2q)^{-3/2}$$

Therefore,

$$fq = -3q \left[1 + \frac{5}{2} q + \frac{5}{2} \cdot \frac{7}{3} q^2 + \dots \right]$$

It follows that

$$\left(\frac{r_{\odot}}{\Delta_{\odot}} \right)^3 = (1 - 2q_{\odot})^{-3/2} = 1 - (fq)_{\odot}$$

and

$$\left(\frac{r_h}{\Delta_h} \right)^3 = (1 - 2q_h)^{-3/2} = 1 - (fq)_h$$

Then (41) can be written

$$(42) \quad \frac{d^2 \bar{r}}{dt^2} = -m_4 k^2 \frac{\bar{r}}{r^3} - \frac{M_{\odot} k^2}{r_{\odot}^3} \left[(fq)_{\odot} \bar{r}_{\odot} + [1 - (fq)_{\odot}] \bar{r} \right] \\ - \frac{m_h k^2}{r_h^3} \left[(\bar{r}_h - \bar{r}) (fq)_h + \bar{r} \right]$$

The units of the astronomical unit, mass of the sun, and ephemeris time can be used.

Many methods of numerical integration can be applied to calculate this orbit. Cowell's method directly integrates the coordinates. Encke's method finds the quantity $\hat{\xi}_1$, where

$$x_1 = x_1 \text{ (ellipse)} + \hat{\xi}_1.$$

After numerical integration, one can combine equation (40) with the residuals into the form of (33) for purposes of orbit improvement.

With $\bar{F} = (F_1, F_2, F_3)$ taken as the right hand side of equation (41), we get the following expressions for some of the partials used in the calculations of w_{1k} and w_{1j} .

$$\begin{aligned} \frac{\partial F_1}{\partial x} &= \frac{m_4 k^2}{r^3} \left[-1 + 3 \frac{x^2}{r^2} \right] + \frac{M_\odot k^2}{\Delta_\odot^3} \left[-1 + 3 \frac{(x_\odot - x)^2}{\Delta_\odot^2} \right] \\ &\quad + \frac{m_h k^2}{\Delta_h^3} \left[-1 + 3 \frac{(x_h - x)^2}{\Delta_h^2} \right] \\ \frac{\partial F_1}{\partial y} &= \frac{m_4 k^2}{r^3} \left[0 + \frac{3xy}{r^2} \right] + \frac{M_\odot k^2}{\Delta_\odot^3} \left[0 + \frac{3(x_\odot - x)(y_\odot - y)}{\Delta_\odot^2} \right] \\ &\quad + \frac{m_h k^2}{\Delta_h^3} \left[\text{similar expression} \right] \end{aligned}$$

etc.

Note that

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_1}{\partial y} = \frac{\partial F_1}{\partial z} = 0$$

since the velocity does not appear in the equations.

It is sometimes convenient to use

$$m_4 \frac{\partial F_1}{\partial m_4} \quad \text{instead of} \quad \frac{\partial F_1}{\partial m_4} \quad . \quad \text{In this case,}$$

$$\delta m_4 \quad \text{is obtained as a ratio} \quad \frac{\delta m_4}{m_4} \quad .$$

V. Orbit Correction Using Elliptic Elements.

A. Rectangular Coordinates

It is possible to obtain a set of orbital elements from the initial position and velocity vectors.

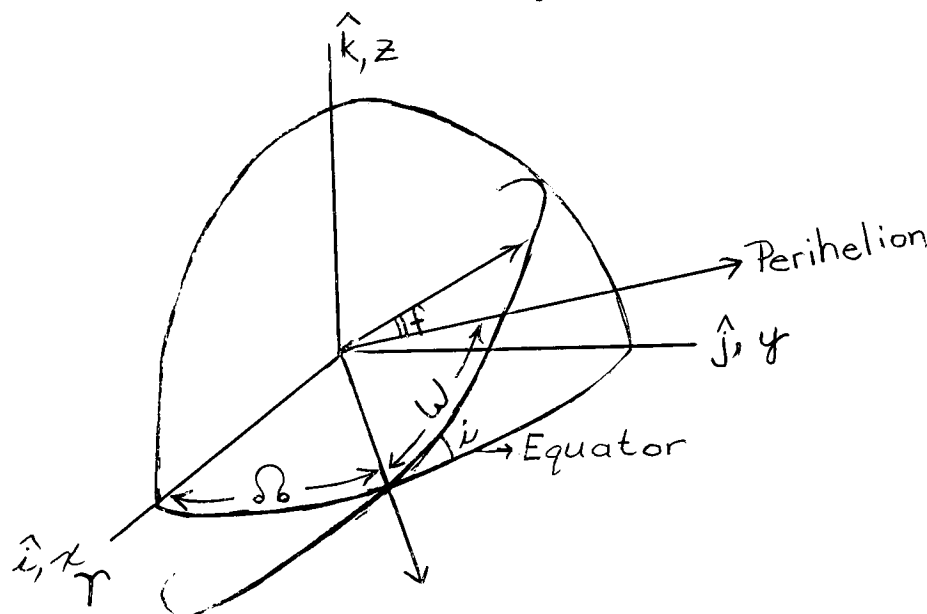


Figure 5.

A set of elements might be

a: semi major axis
 e: eccentricity
 i: inclination
 ω : argument of perihelion
 Ω : longitude of ascending node
 M_0 : mean anomaly at epoch time, t_0 .

As before, the effects of unknown differential variations of the initial conditions are considered. These variations will be represented by

$$\delta M_0, \delta e, \delta a/a, \delta i, \delta \omega, \delta \Omega$$

Replace $\delta i, \delta \omega, \delta \Omega$ by the rotation vector $\delta \bar{\Psi} = (\delta \Psi_x, \delta \Psi_y, \delta \Psi_z)$ where each component represents a rotation about its corresponding axis (x, y, or z). Based on the method of Eckert and Brouwer, the work is done out in full in Herget, pp. 82 f. From considerations developed in the text, it is found that

$$\begin{aligned} \delta \bar{r} = & \delta \bar{\Psi}_x \bar{r} + \frac{\bar{v}}{n} \delta M_0 \\ & + (H\bar{r} + K \frac{\bar{v}}{n}) \delta e \\ & + (\bar{r} + m \frac{\bar{v}}{n}) \frac{\delta a}{a} \end{aligned}$$

H and K are functions of e and E, the eccentric anomaly.

$$m = -\frac{3}{2} k (t - t_0) a^{-3/2}$$

This coefficient of δa , coming from the variation of the mean motion with δa , increases with time. Therefore δa is better determined as time goes on.

$$\delta \bar{\Psi} \times \bar{r} = \hat{i}(\Psi_y z - \Psi_z y) + \hat{j}(\Psi_z x - \Psi_x z) + \hat{k}(\Psi_x y - \Psi_y x)$$

This leads to the following expression for $\delta \bar{r}$ (Herget, p.83).

$$(43) \quad \delta \bar{r} = \begin{bmatrix} 0 & z & -y & x'/n & H_x + K x'/n & x+mx'/n \\ -z & 0 & x & y'/n & H_y + K y'/n & y+my'/n \\ y & -x & 0 & z'/n & H_z + K z'/n & z+mz'/n \end{bmatrix} \begin{bmatrix} \delta \Psi_x \\ \delta \Psi_y \\ \delta \Psi_z \\ \delta m_0 \\ \delta e \\ \delta a/a \end{bmatrix}$$

As before, this expression is substituted into equation (33) and a least squares solution gives values for the variation of the elements.

One difficulty with this development arises in the case of a nearly circular orbit in the x,y plane. In this case, $z' = 0$ and \bar{r}' is very nearly perpendicular to \bar{r} . This implies

$$x' \propto -y$$

$$y' \propto +x$$

The third and fourth rows of the matrix in (43) are proportional and the equations are indeterminate. This necessitates a different formulation.

B. Rectangular Coordinates of the Orbit.

The vectors \hat{P} and \hat{Q} lie in the orbit plane, \hat{P} directed toward perihelion. \hat{R} is perpendicular to the orbital plane. The three vectors form a mutually orthogonal, right handed system of vectors.

Then, choose the following set of variational elements,

$$\begin{aligned}x_1 &= \delta m_0 + \delta s \\x_2 &= \delta p \\x_3 &= \delta q \\x_4 &= e \delta s \\x_5 &= \delta e \\x_6 &= \delta a/a\end{aligned}$$

where δs is a rotation about \hat{R} , δp about \hat{P} , and δq about \hat{Q} .

$$\delta \bar{\Psi} = x_2 \hat{P} + x_3 \hat{Q} + \frac{x_4}{e} \hat{R} \quad .$$

Symbolically, denote $\delta \bar{r}$ by

$$(44) \quad \bar{r} = \begin{bmatrix} c_{1j} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix} \quad \begin{array}{l} i = 1, 2, 3 \\ j = 1, \dots, 6 \end{array}$$

It is necessary to define another set of axes, \hat{r} , \hat{R} , and \hat{u} in order to calculate the c_{1j} .

\hat{r} is a time dependent vector directed along \bar{r} .
 \hat{R} is again the vector perpendicular to the orbital plane.

$$\hat{u} = \frac{\hat{R} \times \bar{r}}{r} .$$

Therefore,

$$\bar{r} = \hat{P} a(\cos E - e) + \hat{Q} b \sin E , \quad b = a \sqrt{1-e^2}$$

$$\frac{\bar{v}}{n} = \frac{\bar{r}'}{n} = \frac{-\hat{P} a \sin E + \hat{Q} b \cos E}{1 - e \cos E}$$

$$\hat{u} = \frac{\hat{Q} (\cos E - e) - \hat{P} \sqrt{1-e^2} \sin E}{1 - e \cos E}$$

$$\hat{r} \cdot \frac{\bar{v}}{n} = \frac{ea^2}{r} \sin E$$

Using methods similar to those in section A, we can find

$$[c_{ij}] .$$

$\delta \bar{\Psi} \times \bar{r} \cdot \hat{r} = 0$. Therefore, $c_{12} = c_{13} = 0$ and the contribution of this term to c_{14} is zero.

$\delta \bar{\Psi} \times \bar{r} \cdot \hat{u} = \frac{r}{e} x_4$. Therefore, $c_{22} = c_{23} = 0$ and the contribution of this term to c_{24} is $\frac{r}{e}$.

$$\delta \bar{\Psi} \times \bar{r} \cdot \hat{R} = b \sin E x_2 - a(\cos E - e) x_3 .$$

Therefore, $c_{32} = b \sin E$, $c_{33} = -a(\cos E - e)$, $c_{34} = 0$.

Similar work ultimately yields the following set of expressions:

$$c_{14} = \frac{-a^2}{r} \sin E$$

$$c_{11} = -e c_{14}$$

$$c_{15} = Hr + K c_{11}^*$$

$$c_{16} = r + m c_{11}$$

$$c_{21} = \frac{a^2}{r} \sqrt{1-e^2} = r - ec_{24}$$

$$c_{24} = \left[\frac{r}{e} - \frac{a^2}{er} \sqrt{1-e^2} \right]$$

*H, k, and m are the same as in Section V, A

The e in the denominator can be removed with some algebraic manipulation.

$$c_{25} = K c_{21}$$

$$c_{26} = m c_{21}$$

$$c_{31} = c_{35} = c_{36} = 0$$

Upon substitution of this matrix into (33) one

$$\frac{1}{\rho} \begin{bmatrix} \leftarrow \bar{\rho} \rightarrow \\ \leftarrow \hat{A} \rightarrow \\ \leftarrow \hat{D} \rightarrow \end{bmatrix} \begin{bmatrix} \\ \\ R \end{bmatrix} \begin{bmatrix} \\ \\ F \end{bmatrix} \begin{bmatrix} \\ \\ c_{1j} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix} = \begin{pmatrix} \Delta \rho \\ \cos \delta \Delta \alpha \\ \Delta \delta \end{pmatrix}$$

$$[F] = \begin{pmatrix} \cos f & -\sin f & 0 \\ \sin f & \cos f & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f = \text{true anomaly}$$

$$[R] = \begin{pmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{pmatrix}, \quad [R] \text{ is derived completely in terms of known angles in Herget, pp. 49 f.}$$

$[R] \cdot [F]$ serve to rotate coordinates from $\hat{r}, \hat{u}, \hat{R}$ to an equatorial frame of reference.

One eliminates $\Delta\rho$ from the equations by ignoring $\bar{\rho}$ in the first matrix.

When the solution is complete, $\delta i, \delta\omega,$ and $\delta\Omega$ are found by

$$\delta i = \cos \omega_0 \delta p - \sin \omega_0 \delta q$$

$$\delta\Omega = \frac{\sin \omega_0 \delta p + \cos \omega_0 \delta q}{\sin i}$$

$$\delta\omega = \delta s - \cos i \delta\Omega$$

When i is small, $\delta\Omega$ is poorly determined.

When e is small, δs is poorly determined.

C. Special Elements for Small e and i .

Trouble with small eccentricity can be avoided by choosing axes not related to perihelion.

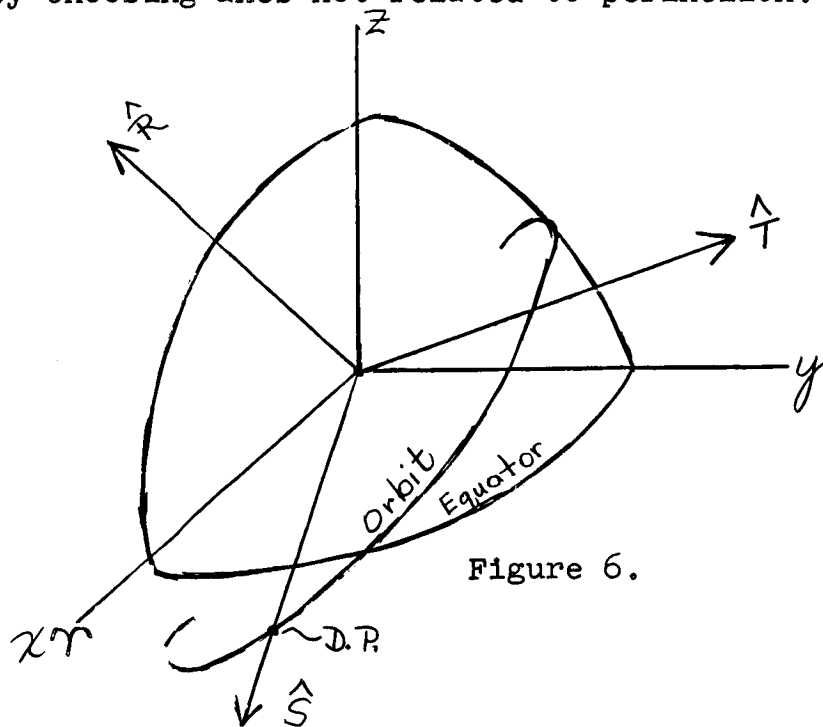


Figure 6.

The point D.P. (departure point) and its associated vector \hat{S} lie in the orbit plane and define an origin of angles. \hat{S} is restricted to motion perpendicular to the plane of the orbit when its variations are calculated. \hat{T} is 90° from \hat{S} in the orbit plane in the direction of motion. $\hat{S} \times \hat{T} = \hat{R}$.

Define

ω = angle from D.P. to perihelion

$u = f + \omega$ = angle from D.P. to \bar{r} .

Corresponding to Kepler's Equation, let

$$\begin{aligned} U &= (M + \omega) = (E + \omega) - e \sin \left[(E + \omega) - \omega \right] \\ &= U_0 + n_0(t - t_0) \\ &= \phi - (X \sin \phi - Y \cos \phi) \\ X &= e \cos \omega, Y = e \sin \omega \\ \phi &= E + \omega \end{aligned}$$

Then

$$\begin{aligned} \frac{r \cos u}{a} &= \cos \phi - X + \frac{Y(X \sin \phi - Y \cos \phi)}{1 + \sqrt{1 - X^2 - Y^2}} \\ \frac{r \sin u}{a} &= \sin \phi - Y - \frac{X(X \sin \phi - Y \cos \phi)}{1 + \sqrt{1 - X^2 - Y^2}} \end{aligned}$$

with

$$\begin{aligned} \bar{r} &= r \cos u \hat{S} + r \sin u \hat{T} \\ e &= \sqrt{X^2 + Y^2} \\ P &= \frac{X\hat{S} + Y\hat{T}}{\sqrt{X^2 + Y^2}} \end{aligned}$$

$$\delta \bar{\Psi} = \Psi_s \hat{S} + \Psi_T \hat{T}$$

one can again find the components of the $[c_{ij}]$ matrix in $\delta \bar{r}$. For example,

$$\delta \bar{\Psi} \times \bar{r} = r \sin u \hat{R} (\delta \Psi_s) - r \cos u \hat{R} (\delta \Psi_T)$$

implies $c_{31} = r \sin u$, $c_{32} = -r \cos u$.

The variational elements are $\delta \Psi_s$, $\delta \Psi_T$, δx , δy , δU_0 , $\delta a/a$.

D. Elliptic Elements with Perturbations.

Define

$$\bar{c} = \bar{r} \times \bar{v} = \sqrt{p} \hat{R}$$

$$\bar{g} = \frac{e}{\sqrt{p}} \hat{Q}$$

\bar{c} and \bar{g} with M are a convenient set of elements to use.

In operator notation, define

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\delta}{\delta t}$$

where $\frac{\partial}{\partial t}$ represents the motion of a particle in an elliptic orbit even if the elements were constant and

$\frac{\delta}{\delta t}$ represents the variation of all constants even if the motion of the particle were frozen fixed.

For example,

$$(45) \quad \begin{aligned} \frac{da}{dt} &= \frac{\partial a}{\partial t} + \frac{\delta a}{\delta t} = \frac{\delta a}{\delta t} ; \frac{\partial a}{\partial t} = 0 \\ \frac{d\bar{r}}{dt} &= \frac{\partial \bar{r}}{\partial t} + \frac{\delta \bar{r}}{\delta t} = \frac{\partial \bar{r}}{\partial t} = \bar{v} ; \frac{\delta \bar{r}}{\delta t} = 0 \end{aligned}$$

Equation (45) says that perturbations cannot instantaneously perturb the radius vector. They have a future but not an immediate influence when they start acting.

$$(46) \quad \begin{aligned} \frac{d\bar{v}}{dt} &= \frac{\partial \bar{v}}{\partial t} + \frac{\delta \bar{v}}{\delta t} = -\frac{\bar{r}}{r^3} + \bar{F} \\ \frac{\partial \bar{v}}{\partial t} &= -\frac{\bar{r}}{r^3} ; \frac{\delta \bar{v}}{\delta t} = \bar{F} . \end{aligned}$$

Therefore, from (46), it is seen that perturbations have an immediate effect on the velocity vector.

Now look at the orbital elements.

$$(47) \quad \frac{d\bar{c}}{dt} = \frac{\delta \bar{c}}{\delta t} = \bar{r} \times \frac{\delta \bar{v}}{\delta t} + \frac{\delta \bar{r}}{\delta t} \times \bar{v} = \bar{r} \times \bar{F} .$$

Formula (47) gives the first variation of \sqrt{p} and \hat{R} . If we dot (47) by \hat{R} ,

$$\hat{R} \cdot \frac{\delta \bar{c}}{\delta t} = \hat{R} \cdot \bar{r} \times \bar{F} = r \hat{u} \cdot \bar{F} = \text{magnitude of change in angular momentum.}$$

Dotting (47) by \hat{r} gives

$$\hat{r} \cdot \frac{\delta \bar{c}}{\delta t} = 0 .$$

The interpretation of this equation is that it is never

possible for \bar{c} to be changed in such a way that it is no longer perpendicular to \bar{r} . That is, any change in the orbit plane must occur perpendicular to \bar{r} and therefore can be expressed as an instantaneous rotation about the radius vector.

$$\hat{u} \cdot \frac{\delta \bar{c}}{\delta t} = \hat{u} \times \bar{r} \cdot \bar{F} = -r \hat{R} \cdot \bar{F}.$$

From this equation it is obvious that if \bar{F} lies in the orbit plane, it will not shift the plane at all.

For the variation of \bar{G} , one writes

$$(48) \quad \frac{\delta \bar{G}}{\delta t} = \left(1 + \frac{r}{p}\right) \bar{F} - \frac{[\bar{r} \cdot \bar{F}]}{rp} \bar{r} - \frac{2r}{p} \left[\hat{R} \cdot \bar{F} \right] \hat{R}$$

For purposes of notation,

$$\frac{\delta \bar{G}}{\delta t} = \bar{K} - \frac{2r}{p} \left[\hat{R} \cdot \bar{F} \right] \hat{R}.$$

To find the variation of M , consider first the variation of a . From the energy integral,

$$2\bar{v} \cdot \frac{\delta \bar{v}}{\delta t} = \frac{-2}{r^2} \frac{\delta \bar{r}}{\delta t} + \frac{1}{a^2} \frac{\delta a}{\delta t}.$$

Therefore,

$$\frac{\delta a}{\delta t} = 2a^2 \bar{v} \cdot \bar{F}$$

and

$$\frac{\delta n}{\delta t} = -\frac{3k}{\sqrt{a}} \bar{v} \cdot \bar{F}.$$

Then,

$$\frac{dm}{dt} = \frac{\partial m}{\partial t} + \frac{\delta m}{\delta t} = n + \frac{\delta m_0}{\delta t}$$

Therefore,

$$M = m_0 + \int \frac{\delta m_0}{\delta t} dt + \int \left[n_0 + \int \frac{\delta n}{\delta t} dt \right] dt =$$

$$m_0 + \Delta M + n_0(t-t_0) + \iiint -\frac{3k}{\sqrt{a}} \bar{v} \cdot \bar{F} dt dt$$

where

$$\frac{\delta m_0}{\delta t} = \frac{\sqrt{1-e^2}}{|\bar{a}|} \hat{P} \cdot \bar{K} - \frac{2}{\sqrt{a}} (\bar{r} \cdot \bar{F})$$

If the eccentricity is small, there will be difficulty in the calculation of \hat{Q} and $\frac{\delta m_0}{\delta t}$. (See an article by Herget, Astronomical Journal, 57, 1952).

A possible way to avoid this deficiency is to go back to the set of elements defined in Section V, C.

Start with

$$U = U_0 + n_0(t-t_0) + \int \frac{\delta U_0}{\delta t} dt + \iiint \frac{\delta n}{\delta t} dt dt$$

$$X = X_0 + \int \left\{ \left[R \cdot \frac{\delta \bar{c}}{\delta t} \right] (\bar{v} \cdot \hat{T}) + |\bar{c}| (\bar{F} \cdot \hat{T}) \right\} dt$$

$$\bar{v} \cdot \hat{T} = \frac{x + \cos u}{\sqrt{p}}$$

$$Y = Y_0 + \int \left\{ -\left[\hat{R} \cdot \frac{\delta \bar{c}}{\delta t} \right] (\bar{v} \cdot \hat{S}) - |\bar{c}| (\bar{F} \cdot \hat{S}) \right\} dt$$

Letting \bar{Q} represent a finite rotation about \hat{S} , \hat{T} , \hat{R} from position at t_0 to the instantaneous position, we can write

$$\hat{U} = \hat{U}_0 + \frac{2}{1 + Q^2} \left[\bar{Q} \times U_0 + \bar{Q} \times (\bar{Q} \times \hat{U}_0) \right]$$

where \hat{U} represents \hat{S} and \hat{T} .

$$\bar{Q} = \int \frac{\delta \bar{Q}}{\delta t} dt$$

and

$$\frac{\delta \bar{Q}}{\delta t} = \frac{\hat{R} \cdot \bar{F}}{Jp} \frac{(1+Q^2)}{4} (\bar{r} + \bar{r}_0) \quad .$$

VI. The Parabolic Orbit.

A. Parameters Defined.

Consider the relation

$$(49) \quad r^2 dv = k \sqrt{p} dt$$

from which Kepler's equation is derived.

$$M = E - e \sin E.$$

As $e \rightarrow 1$, $M \rightarrow E^3/e! + \dots$. The relation is cubic. This implies the need to redefine Kepler's equation for $e \geq 1.0$.

For a hyperbola, $e > 1.0$,

$$M = v(t-T) = -F + e \sinh F. \quad (\text{Herget, p. 34}).$$

For the parabola, $e = 1.0$, the following transformation is used. (Herget, p. 32).

Define $q = p/2$, $p = \text{semi-latus rectum}$.

$$\text{Then } r = \frac{p}{1 + e \cos f} = \frac{2q}{1 + \cos f}$$

$$= q \sec^2 \frac{f}{2} = q (1 + \tan^2 \frac{f}{2})$$

where $f = \text{true anomaly}$.

Substitution into equation (49), gives

$$\frac{k \, dt}{\sqrt{2} \, q^{3/2}} = \sec^2 \frac{f}{2} (1 + \tan^2 \frac{f}{2}) \, d(\frac{1}{2}f)$$

Integrating, from T to the variable upper limit, t , gives

$$\frac{k(t-T)}{\sqrt{2} \, q^{3/2}} = \tan(\frac{1}{2}f) + \frac{1}{3} \tan^3(\frac{1}{2}f) .$$

B. Position Determination - Gauss for Nearby Parabolic Orbit. (Herget, pp. 35 ff).

Define

$$a = \sqrt{\frac{1 + 9e}{10}} , \quad b = \frac{5(1-e)}{1+9e} , \quad c = \sqrt{\frac{5(1+e)}{1+9e}}$$

$$(50) \quad A = \frac{15 (E - \sin E)}{9E + \sin E} = b \tan^2(\frac{1}{2}w) .$$

$$cC \tan \frac{1}{2}w = \tan \frac{1}{2}f$$

$$(51) \quad Ba \frac{k(t-T)}{\sqrt{2} q^{3/2}} = \tan \frac{1}{2}w + \frac{1}{3} \tan^3 \frac{1}{2}w$$

$$r = q D(1 + \tan^2 f/2)$$

$$(52) \quad r \cos f = q D(1 - \tan^2 f/2)$$

$$r \sin f = q D(2 \tan f/2)$$

Therefore, B, C, and D can all be written as functions of A. When $A = 0$, $B = C = D = 1$. Tables of values of B, C, D for a given value of A are found in the Appendix of the text, for elliptic and hyperbolic cases.

One may find the solution by starting with $B = 1$ in (51). This gives w which in turn yields a value for A in (50). From the tables, this A gives a new value for B , etc.

One can also find other orbital quantities from the relations in (52).

C. Orbit Determination - Lambert (Herget, pp. 65 ff).

Define

$$2g = E_1 - E_j = c - d$$

$$2h = c + d$$

$$\text{and } \cosh = e \cos \left(\frac{E_1 + E_j}{2} \right)$$

$$\text{Then } r_1 + r_j = 2a(1 - \cos g \cosh) = a(2 - \cos c - \cos d) \quad .$$

For the vector $\bar{S} = \bar{r}_1 - \bar{r}_j$ (the chord joining any two positions in the orbit) we have

$$S^2 = 4a^2 (1 - \cos g \cosh)^2 - 4a^2 (\cos g - \cosh)^2$$

[the minus sign in the second half of the right hand side is missing in the text]

$$\begin{aligned} S^2 &= (2a \sin g \sinh)^2 \\ &= a^2 (\cos d - \cos c)^2 \end{aligned}$$

$$r_1 + r_j + S = 4 \sin^2(\tfrac{1}{2}c)$$

$$(53) \quad r_1 + r_j - S = 4 \sin^2(\tfrac{1}{2}d) .$$

Kepler's equation gives the dynamical conditions, in the form

$$\begin{aligned} \frac{k(t_j - t_1)}{a^{3/2}} &= 2g - 2 \sin g \cosh \\ &= (c - \sin c) - (d - \sin d) . \end{aligned}$$

Then

$$\begin{aligned} 6k(t_j - t_1) &= \frac{3}{4} \left(\frac{c - \sin c}{\sin^3(\tfrac{1}{2}c)} \right) (4a \sin^2(\tfrac{1}{2}c))^{3/2} \\ &\quad - \frac{3}{4} \left(\frac{d - \sin d}{\sin^3(\tfrac{1}{2}d)} \right) (4a \sin^2(\tfrac{1}{2}d))^{3/2} \\ (54) \quad \text{or} \quad 6k(t_j - t_1) &= Q(c) [r_1 + r_j + S]^{3/2} - Q(d) [r_1 + r_j - S]^{3/2} . \end{aligned}$$

This equation can be expanded (Herget, p. 66) into a series. Then taking the limit as $a \rightarrow \infty$, one gets Euler's equation (first derived by Newton).

Equations (53) and (54) represent the geometrical and dynamical conditions of the orbit. Recall the equation of Gauss's method,

$$(55) \quad c_1 \bar{\rho}_1 - \bar{\rho}_2 + c_3 \bar{\rho}_3 = c_1 \bar{R}_1 - \bar{R}_2 + c_3 \bar{R}_3 = \bar{V}.$$

Define

$$\frac{\bar{\rho}_2 \times \bar{V}}{|\bar{\rho}_2 \times \bar{V}|} = \hat{W}.$$

$$\hat{W} \times \hat{\rho}_2 = \hat{u}$$

Then $\hat{V} \cdot \hat{\rho}_2 \times \hat{u} = 0$. ($\hat{\rho}_2$ is taken as the epoch time).
Dot (55) by $\hat{\rho}_2 \times \hat{u}$.

$$\text{Then, } c_1 \rho_1 (\hat{\rho}_1 \cdot \hat{\rho}_2 \times \hat{u}) + c_3 \rho_3 (\hat{\rho}_3 \cdot \hat{\rho}_2 \times \hat{u}) = 0$$

Therefore $\rho_3 = M \rho_1$ where

$$M = \frac{c_1 (\hat{\rho}_1 \cdot \hat{\rho}_2 \times \hat{u})}{c_3 (\hat{\rho}_3 \cdot \hat{\rho}_2 \times \hat{u})} \quad \text{and is perfectly rigorous.}$$

If the c 's are known, this determines M . The development is found in Herget, pp. 67 f.

One important difficulty is overcome when a parabolic orbit is determined. Since $e = 1$, there remain only 5 unknowns to be calculated. Therefore, 3 observations, even if they lie on a great circle, are sufficient to determine the orbit.

D. Differential Corrections to Parabolic Orbit.

From previous considerations, it is known that

$$\bar{r} = q \hat{P} (1 - \tan^2 f/2) + 2q \hat{Q} \tan f/2 .$$

$$r_1' = \frac{2q \hat{Q} - 2q \hat{P} \tan f/2}{\sqrt{2} q^{3/2} (1 + \tan^2 f/2)} .$$

The elements of the orbit can be specified by

q, T, \hat{P}, \hat{Q} where $e = 1$. Therefore we can use the following set of differential variations.

$$\delta\psi_x, \delta\psi_y, \delta\psi_z, k\delta T, \delta q/q . \quad \delta e = 0 .$$

With respect to an equatorial frame of reference, using equations analagous to those in Section V, we obtain

$$\delta\bar{r} = \begin{bmatrix} 0 & z & -y & -x' & x - \frac{3}{2} k(t-T) x' \\ -z & 0 & +x & -y' & y - \frac{3}{2} k(t-T) y' \\ y & -x & 0 & -z' & z - \frac{3}{2} k(t-T) z' \end{bmatrix} \begin{bmatrix} \delta\psi_x \\ \delta\psi_y \\ \delta\psi_z \\ k\delta T \\ \delta q/q \end{bmatrix}$$

The inclusion of δe , letting a parabola go into an ellipse or hyperbola, is discussed in an article by Paul Herget, Astronomical Journal, 48, 105, (1940).

CALCULUS OF VARIATIONS AND OPTIMUM CONTROL THEORY

by

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11.5.5

I. Introduction.

The calculus of variations and optimum control theory, along with certain associated computational methods, will be presented in parallel format to show the basic similarities in spite of what may superficially seem to be glaring differences. The two theories together form one theory, with separate vocabularies arising from usage current to its era of development.

Consider the following problem in classical calculus of variations, the well known bead on the frictionless wire falling under the influence of gravity or brachistochrone:

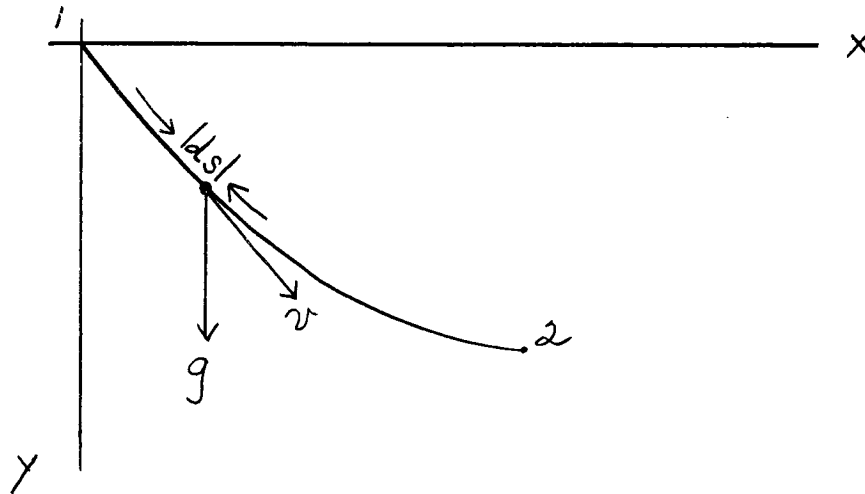


Fig. 1

Find the path of least time between points 1 and 2 for a bead of mass m sliding along the wire under the influence of gravity alone. The time required for descent is

$$T = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{ds}{\sqrt{2gy}} = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dy, \quad (1.1)$$

where the last integral is written for a curve $y=y(x)$, $x_1 \leq x \leq x_2$. Restated: Of all arcs joining the points 1 and 2, find the arc for which $T=\min$.

Consider now the modern brachistochrone problem, that of finding the path of least time between two points for a rocket under the influence of gravity and a thrust force with variable direction but with constant magnitude.

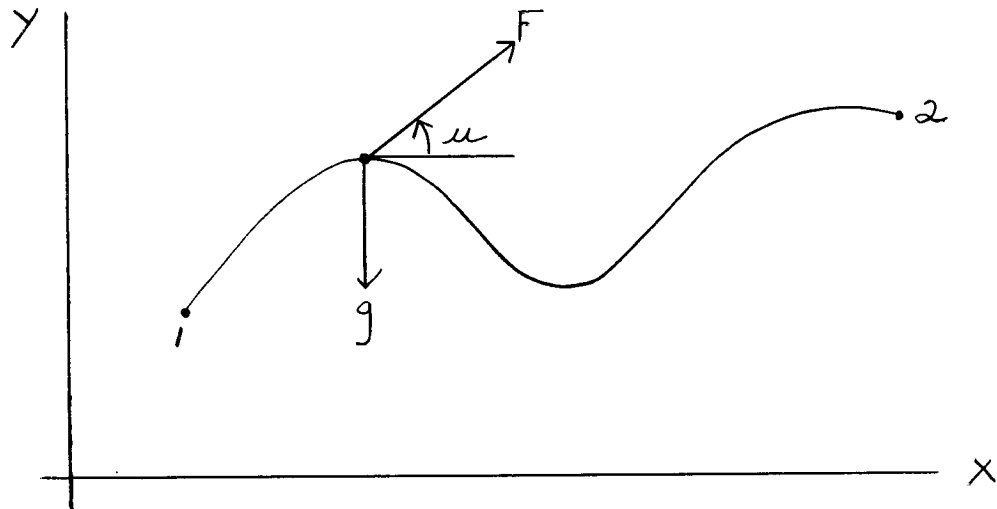


Fig. 2

An additional constraint is imposed: The slope of the optimal path is to have fixed values at 1 and 2. This is a problem in optimum control theory. Mathematically formulated in terms of the variables shown in Fig. 2, for a rocket of mass 1:

The differential equations are $\ddot{x} = F \cos u$

$$\ddot{y} = F \sin u - g.$$

The end conditions are $x(0), y(0), \dot{x}(0), \dot{y}(0)$ $\left\{ \begin{array}{l} \text{fixed,} \\ x(T), y(T), \dot{x}(T), \dot{y}(T) \end{array} \right\}$ (1.2)

and the problem is to make T a minimum.

This control problem is, in fact, the classically formulated Problem of Mayor. One speaks of the variables x, \dot{x}, y, \dot{y} as the state variables, and of the function $u(t)$ as the control variable. We wish to choose $u(t)$ so that we go from point 1 to point 2 in the least time.

Let us rewrite the last problem in a more convenient form. Let

$$x^1 = x, x^2 = y, x^3 = \dot{x}, x^4 = \dot{y}. \quad (1.3)$$

Then the problem is: Differential equations

$$\dot{x}^1 = x^3, \dot{x}^2 = x^4, \dot{x}^3 = F \cos u, \dot{x}^4 = F \sin u - g, \text{ with } x^1(0) \text{ fixed, } x^1(T) \text{ fixed } (i=1,2,3,4); T=\min. \quad (1.4)$$

This type of problem can also be written in the form of the general Problem of Bolza:

Given $\dot{x}^i = f^i(t, x, u)$, ($i=1, \dots, n$), a set of differential or algebraic equations, find among the class of arcs satisfying some end point conditions, say $x^1(0)$ fixed, and perhaps $x^1(T)$ on a line or surface in x^1 space, the functions $x^1(t)$ and the control $u(t)$, $0 \leq t \leq T$, for which

$$g(t) + \int_0^T f(t, x, u) dt = \min. \quad (1.5)$$

It is to be understood that the symbols x and u represent vectors with, in the case of x , n components.

Among the topics we could consider regarding are these problems:

1. Properties of solutions,
2. Construction of solutions,
3. Existence of solutions,
4. Sufficiency conditions.

In this series of lectures we will consider only Topic 1, which includes complete discussions of the necessary conditions which must be satisfied by solutions of the above-formulated problems.

II. Minimum of a Function of n Variables.

Before studying the problem of minimizing a functional such as (1.1), let us consider the problem of minimizing a function of n variables. As an example, consider

$$f(x, y) = \min.$$

The first order necessary conditions that must be satisfied are

$$f_{\dot{x}} = 0, f_{\dot{y}} = 0, \text{ where, for example, } f_x = \frac{\partial f}{\partial x}, \text{ etc.;} \quad (2.1)$$

and the second order test is

$$f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \geq 0. \quad (2.2)$$

Of course, these conditions guarantee only that a point is a local minimum. Since there is no global test for the absolute minimum, we usually must find all the points satisfying (2.1) and (2.2) and then test to ascertain the absolute minimum.

In the more general case of a function of n variables, $f(x^1, x^2, x^3, \dots, x^n)$, we write the necessary conditions analogous to (2.1) and (2.2) as

$$\begin{aligned} f_{x^i}(x_0) &= 0 & i &= 1, 2, \dots, n \\ f_{x^i x^j}(x_0) h^i h^j &\geq 0 & \text{for all } h, \end{aligned} \quad (2.3)$$

which must be satisfied for all points x_0 which are minima. In (2.3), the usual summation convention has been adopted. (2.3)₁ can be interpreted as the condition that $\text{grad } f = 0$. To see this, let

$$\phi(t) = f(x_0 + th) \geq f(x_0) = \phi(0)$$

if x_0 is a minimum point. Thus $\phi'(0) = 0$ and $\phi''(0) \geq 0$ for such a point, a condition that must be true for all h . Thus it follows that

$$0 = \phi'(0) = f'(x, h) = f_{x^i}(x_0) h^i, \quad (2.4)$$

which is identical to (2.3)₁. (2.4) is also sometimes called the differential of f at x_0 , the first variation of f at x_0 and the directional derivative of f at x_0 in

the direction h . $(2.3)_2$ is obtained from the latter condition on $\phi(t)$,

$$0 \leq \phi'' = f''(x_0, h) = \frac{d^2 f}{dt^2} (x_0 + th) \Big|_{t=0} = f_{x^i x^j}(x_0) h^i h^j.$$

As an example, let us find the shortest distance from a point P , say $(3,4)$, to the circle centered at the origin, radius 1.

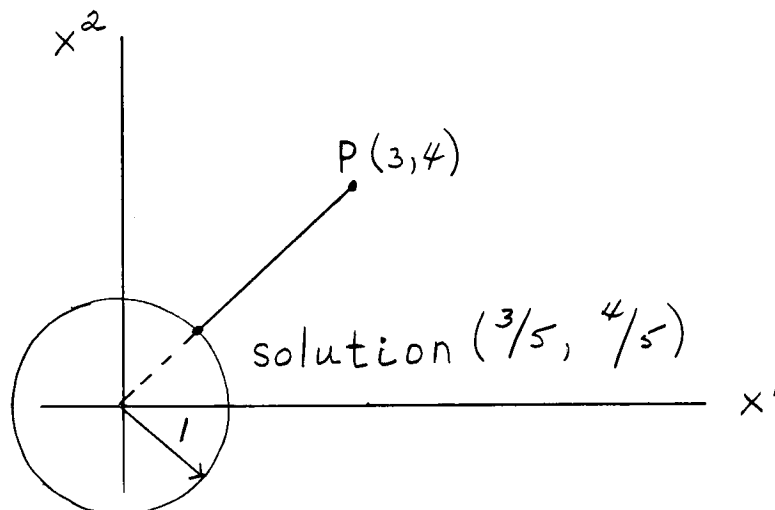


Fig. 3

Minimize $\left[(x^1 - 3)^2 + (x^2 - 4)^2 \right]^{\frac{1}{2}}$, or simply

$$f_0(x) = \frac{1}{2} \left[(x^1 - 3)^2 + (x^2 - 4)^2 \right]$$

subject to the constraint

$$f_1(x) = \frac{1}{2} \left[1 - (x^1)^2 - (x^2)^2 \right] \geq 0,$$

where the inequality constraint has been imposed for generality. Computing the directional derivative,

$$f'_0(x_0, h) = -\frac{4}{5}(3h^1 + 4h^2) \equiv k_0$$

$$f'_1(x_0, h) = -3(3h^1 + 4h^2) \equiv k_1,$$

we observe the relation

$$k_0 - \frac{4}{5} k_1 = 0, \quad (2.5)$$

which is the "multiplier rule" for this very simple case.

We note in this example that inside the circle $f'_1 < 0$, while $f'_1 > 0$ outside the circle. Now (2.5) requires that k_0 and k_1 are both positive, both negative, or both vanished. Thus if $f_0(x_0)$ is a minimum, $k_0 > 0$; hence $k_1 > 0$. Calling K the vector with components k_ρ ($\rho = 0, 1$), and \bar{K} the class of inadmissible vectors, i.e., all \bar{k}_ρ such that $k_0 < 0$, $k_1 \geq 0$, the multiplier rule (2.5) can be restated in a disguised form:

No \bar{k} in \bar{K} is in K .

This is the form of the multiplier rule found in modern texts such as Pontryagin [1]*.

One further example is the problem of finding the shortest distance between the circle, Fig. 3, and a point P which is constrained to lie on or above the line $3x^3 + 4x^4 - 25 = 0$. The mathematical formulation is

$$f_0(x) = \frac{1}{2}[(x^1 - x^3)^2 + (x^2 - x^4)^2] = \min$$

subject to

*Numbers in square brackets refer to Bibliography at the end of the paper.

$$f_1(x) = \frac{1}{2}[1 - (x^1)^2 - (x^2)^2] \geq 0$$

$$f_2(x) = 3x^3 + 4x^4 - 25 \geq 0.$$

To solve this problem, we look at all vectors $K = (k_1, k_2, k_3)$, where $k_1 = f'_1(x, h)$, with h an arbitrary vector. Here $x = (x^1, x^2, x^3, x^4)$; $x_0 = (3/5, 4/5, 3, 4)$ is the known solution. \bar{K} is \bar{K}_P such that $\bar{k}_1 \geq 0$, $\bar{k}_2 \geq 0$, and $\bar{k}_0 \leq 0$. This can be seen by considering how the functions f_1 and f_2 change as the point P and the terminal point at the circle move, as in the previous example. For this case,

$$k_0 = f'_0(x_0, h) = -\frac{4}{5}[3(h^1 - h^3) + 4(h^2 - h^4)]$$

$$k_1 = f'_1(x_0, h) = -(3h^1 + 4h^2)$$

$$k_2 = f'_2(x_0, h) = (3h^1 + 4h^2).$$

Thus the multiplier rule is simply

$$k_0 - \frac{4}{5} k_1 - \frac{4}{5} k_2 = 0.$$

If we write $F = f_0 - \frac{4}{5} f_1 - \frac{4}{5} f_2$, then the multiplier rule is

$$F'(x_0, h) = k_0 - \frac{4}{5} k_1 - \frac{4}{5} k_2 = 0,$$

which is equivalent to

$$F_{x^1} = 0 \quad \text{at } x_0.$$

Exercise: Solve $f_0 = \frac{1}{2}[(x^1 - x^3)^2 + (x^2 - x^4)^2] = \min$

with constraints

$$f_1(x) = \frac{1}{2}[1 - (x^1)^2 - (x^2)^2] \geq 0$$

$$f_2(x) = x^3 + x^4 - 7 \geq 0$$

$$f_3(x) = x^3 + 2x^4 - 11 \geq 0.$$

Let us consider the theory of minima of functions of n variables in more detail now that we have an idea of what must be observed, in view of the simple examples given above. Because every problem that is to be solved numerically must be discretized, i.e., reduced to a problem given in terms of functions of n variables, it is important to have a good grasp of the theory before proceeding to more advanced topics.

For the function $f(x) = f(x^1, x^2, \dots, x^n)$, the level surfaces are those for which $f(x) = \text{constant}$. As we know, the vector normal to a level surface, i.e., the vector in the direction of greatest rate of change of f , is $\text{grad } f$ and has the components $f_{x^j} \equiv \frac{\partial f}{\partial x^j}$ ($j=1, \dots, n$).

The rate of change in any other direction $h = (h^1, h^2, \dots, h^n)$ is then $\text{grad } f \cdot h$ or

$$\begin{aligned} \text{grad } f \cdot h &\equiv f'(x_0, h) = \left. \frac{d}{dt} f(x_0 + th) \right|_{t=0} \\ &= \left. \frac{\partial f}{\partial x^1} \right|_{x_0} h^1 + \left. \frac{\partial f}{\partial x^2} \right|_{x_0} h^2 + \dots + \left. \frac{\partial f}{\partial x^n} \right|_{x_0} h^n, \end{aligned}$$

where $x_0 + th \equiv (x_0^1 + th^1, x_0^2 + th^2, \dots, x_0^n + th^n)$.

Thus we write f prime,

$$f'(x_0, h) = f_{x^1}(x_0) h^1 = \text{grad } f \cdot h = (\text{grad } f, h),$$

as the directional derivative of f in the direction h .

If the level curve is as shown in Fig. 4, and assuming $\text{grad } f \neq 0$, then for h_1 , $f'(x_0, h) > 0$, for h_2 , $f'(x_0, h) < 0$, and for h_3 , $f'(x_0, h) = 0$, since $\text{grad } f$ is normal to the level surface.

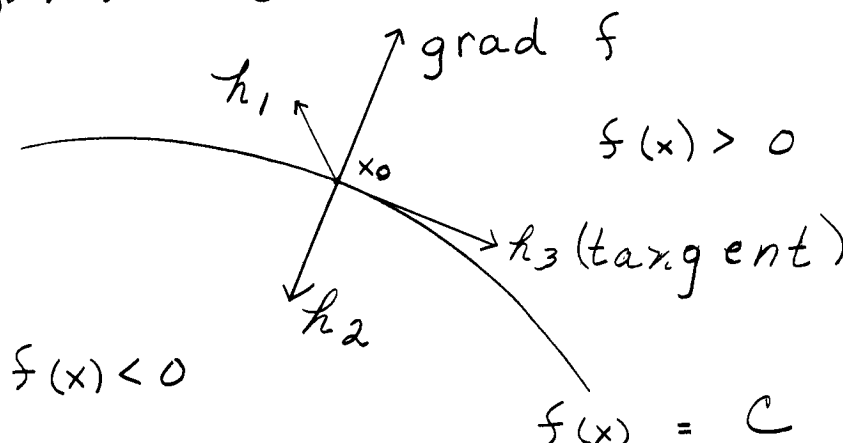


Fig. 4

The directional derivative can be modified by specifying h to lie tangent to some curve $x(t)$ that intersects the curve $f(x)=C$, i.e., we require that $x(0)=x_0$, $\dot{x}(0)=h$. Then

$$\begin{aligned} f'(x_0, h) &= \left. \frac{d}{dt} f(x(t)) \right|_{t=0} \\ &= f_{x^1}(x_0) \dot{x}^1(0), \end{aligned}$$

where $\dot{x}^1(0)$ has replaced h^1 .

The economy of the notation introduced here enables

us to write Taylor's Theorem as follows:

For one variable $f(x) = f(x_0) + f'(x_0) (x-x_0) +$

$$\frac{1}{2} f''(x_0) (x-x_0)^2 + \dots,$$

and

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2} f''(x_0)h^2 + \dots$$

For n variables we write Taylor's Theorem as

$$f(x_0+h) = f(x_0) + f'(x_0, h) + \frac{1}{2} f''(x_0, h) + \dots,$$

$$\text{where } f'(x_0, h) = f_{x^1} h^1$$

$$f''(x_0, h) = f_{x^1 x^j} h^1 h^j.$$

Suppose that x_0 is the solution of the problem

$$f(x) = \min.$$

How do the level surfaces look near x_0 ? From the expansion

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0, x-x_0) + \dots = \text{constant},$$

since $f'(x_0) = 0$. In two dimensions,

$$f(x, y) = f(x_0, y_0) + f_{xx} (x-x_0)^2 + 2f_{x,y} (x-x_0) (y-y_0) +$$

$$f_{yy} (y-y_0)^2 = \text{constant}.$$

Truncation of the series at the second order terms shows

that near the minimum point the level surfaces are ellipses.

For a problem with constraints, the classical procedure is to introduce Lagrange multipliers, e.g., in the problem

$$f(x) = \min$$

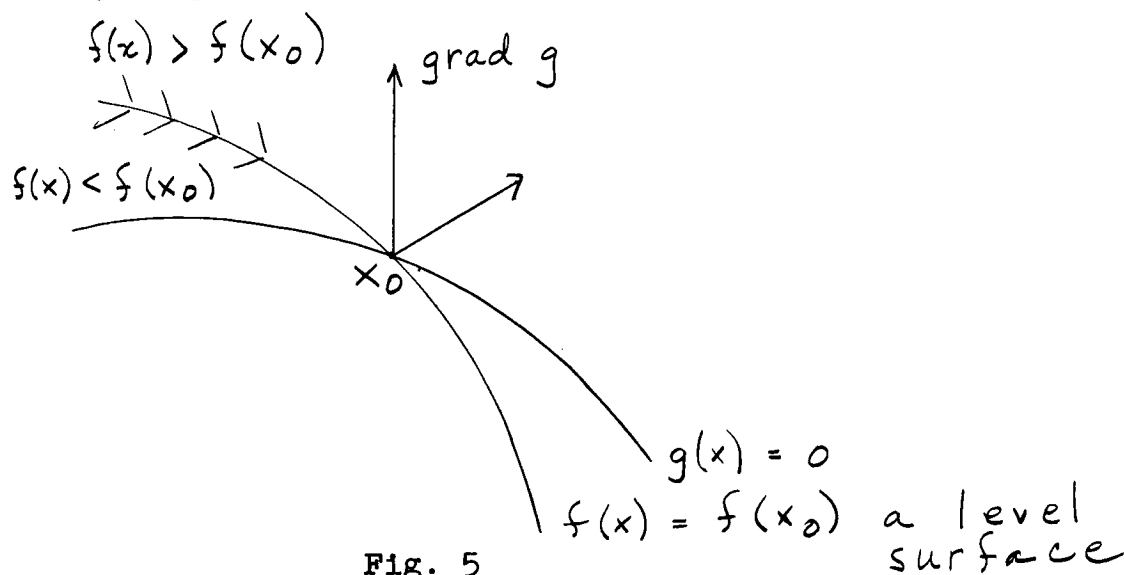
$$\text{subject to } g(x) = 0.$$

Form the function $F(x) = f(x) - \lambda g(x)$. We will choose λ such that

$$F_{x_1}(x_0) = \text{grad } F|_{x_0} = 0,$$

where x_0 is the solution point. There is a unique λ provided $\text{grad } g \neq 0$. To see this graphically, consider Fig. 5. It is clear that in order for a solution to exist, the curves $f(x) = f(x_0)$ and $g(x) = 0$ must not cross but must be tangent at the solution point, for only in that case is it possible to choose a λ so that

$$\text{grad } f = \lambda \text{ grad } g.$$



In the above problem we can accept either $\lambda > 0$ or $\lambda < 0$. However, for a problem with an inequality constraint, say,

$$\begin{aligned} f(x) &= \min \\ g(x) &\geq 0, \end{aligned}$$

with solution x_0 , it can be shown by similar graphical arguments that in order for $\text{grad } F = \text{grad } f - \lambda \text{ grad } g = 0$, λ must be non-negative.

To summarize, we state the following without proof:

Theorem: For the problem

$$f(x) = \min$$

$$\text{and Case I: } g_1(x) = 0$$

$$g_2(x) = 0$$

$$\text{or Case II: } g_1(x) = 0$$

$$g_2(x) \geq 0$$

$$\text{or Case III: } g_1(x) \leq 0$$

$$g_2(x) \geq 0,$$

if x_0 is a solution, i.e., $g_1(x_0) = 0$, $g_2(x_0) = 0$, then there exist multipliers λ_1 and λ_2 such that, when we set

$$F = f + \lambda_1 g_1 + \lambda_2 g_2,$$

$$F_{x^1} = \text{grad } F = 0 \text{ at } x_0.$$

For Case II: $\lambda_2 \geq 0$;
 for Case III: $\lambda_2 \geq 0$; $\lambda_3 \geq 0$.

These results will now be interpreted in terms of the vectors K and \bar{K} introduced earlier. If we write

$$\begin{aligned} k_0 &= f'(x_0, h) \\ k_1 &= g_1'(x_0, h) \\ k_2 &= g_2'(x_0, h), \end{aligned}$$

then for $k_1 \geq 0$ and $k_2 \geq 0$, we must have $k_0 \geq 0$.

Equivalent to the above Theorem is the following:

Theorem:

Let K be all vectors $k = (k_0, k_1, k_2)$ and let \bar{K} be all vectors $\bar{k} = (\bar{k}_0, \bar{k}_1, \bar{k}_2)$ such that $\bar{k}_0 < 0$, $\bar{k}_1 \geq 0$, and $\bar{k}_2 \geq 0$.

Then no vector \bar{k} in \bar{K} is in K .

More generally, for the problem

$$f(x) = \min$$

$$\begin{aligned} \text{subject to } g_\alpha(x) &= 0 & (\alpha = 1, \dots, m') \\ g_\beta(x) &\geq 0 & (\beta = m'+1, \dots, m), \end{aligned}$$

and if x_0 is a solution, i.e.,

$$\begin{aligned} g_\alpha(x_0) &= 0 & (\alpha = 1, \dots, m') \\ g_{\beta'}(x_0) &= 0 & (\beta' = m'+1, \dots, m'') \\ g_{\beta''}(x_0) &> 0 & (\beta'' = m''+1, \dots, m), \end{aligned}$$

then we have the multiplier rule:

There exist multipliers $\lambda_0 \geq 0$, $\lambda_1, \dots, \lambda_m$ such that

$$1) \quad \lambda_0 \geq 0 \text{ and}$$

$$2) \quad \lambda_0'' = 0.$$

3) The function $F = \lambda_0 f - \lambda_\gamma g_\gamma$ ($\gamma=1, \dots, m$) has the property that $\text{grad } F=0$ at x_0 . If the matrix

$$\left(\frac{\partial g_\sigma(x_0)}{\partial x^1} \right) \quad (\sigma=1, \dots, m'')$$

has the rank m'' , then $\lambda_0 > 0$ and can be chosen $\lambda_0=1$.

If so chosen, the multipliers are unique.

III. Classical Calculus of Variations.

Let us now return to classical theory and derive the necessary conditions for a minimum in a general form. Problem (1.1), the brachistochrone problem, can be written

$$J(y) = \int_{x_1}^{x_2} \frac{\sqrt{1+y^2}}{\sqrt{y}} dx = \min. \quad (3.1)$$

Where we now write $y: y(x)$, $x_1 \leq x \leq x_2$, $[x_1, y(x_1)]$, $[x_2, y(x_2)]$ are held fast, and unessential constants have been ignored. Another source of problems is that of the minimization of the area of a surface of revolution, the generator of which passes through any two points 1 and 2, Fig. 6.

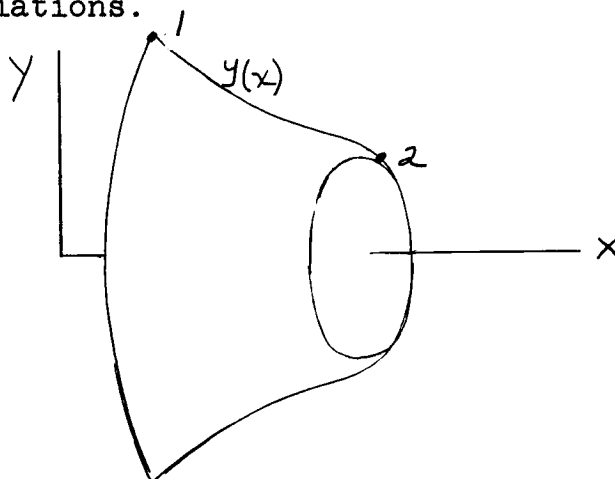


Fig. 6

The functional to be minimized is

$$J(y) = \int_{x_1}^{x_2} 2\pi y ds = \int_{x_1}^{x_2} 2\pi y \sqrt{1+y'^2} dx = \min, \quad (3.2)$$

where the factor 2π may be dropped, since it is unimportant. A broad spectrum of typical problems is covered by the following forms of functionals:

$$J(y) = \int_{x_1}^{x_2} y^2 \sqrt{1+y'^2} dx = \min \quad (r \text{ is real})$$

$$J(y) = \int_{x_1}^{x_2} (y'^2 - y^2) dx = \min \quad \text{and}$$

$$J(y) = \int_{x_1}^{x_2} \sqrt{1-y'^2} dx = \min.$$

In the general form, the fixed end point problem is written:

Determine Y : $y(x)$, $x_1 \leq x \leq x_2$, with

$[x_1, y(x_1)]$, $[x_2, y(x_2)]$ held fast such that

$$J(y) = \int_{x_1}^{x_2} f[x, y(x), y'(x)] dx = \min.$$

If the minimizing arc is $y_0: y_0(x)$, $x_1 \leq x \leq x_2$, then we have the

Main Theorem: 1) $f - y'f_{y'}$, is continuous along y_0 and

$$\frac{d}{dx} (f - y'f_{y'}) = f_x \text{ on } y_0; \quad (3.3)$$

$f_{y'}$, is continuous along y_0 and

$$\frac{d}{dx} f_{y'} = f_y \text{ on } y_0, \quad (3.4)$$

which are the Euler equations, and

2) For admissible arcs with $x, y(x), y'(x)$ in the region of definition of $f(x, y, y')$, R

$$E(x, y_0(x), y'_0(x), Y') \geq 0, \quad (3.5)$$

where $E(x, y, y', Y') = f(x, y, Y') - f(x, y, y') - (Y' - y')f_{y'}(x, y, y')$.

(3.5) is the Weierstrass Condition.

Before proceeding with the proof of the above Theorem, let us consider a few examples.

If $f = y'^2 - y^2$, then $f_{y'} = 2y'$, i.e., there can be no broken corners on y_0 . Since $f_x = 0$, we have the condition from 1) above

$$f - y'f_{y'} = -y'^2 - y^2 = \text{constant},$$

and since $f_y = -2y$, the Euler equation is

$$y'' + y = 0,$$

which has the solution

$$y = a \cos x + b \sin x.$$

$$\text{If } f = y^2 \sqrt{1+y'^2}, \quad f_{y'} = \frac{y^2 y'}{\sqrt{1+y'^2}}.$$

$$f - y' f_{y'} = y^2 \cdot \frac{1}{\sqrt{1+y'^2}}.$$

The Euler equation is then integrated once to give a conservation principle:

$$\frac{y^2}{\sqrt{1+y'^2}} = \text{constant}.$$

Exercise: Show in the above example that for $r=1$,

$$y = b \cosh \frac{x-a}{b}.$$

Discuss the cases $r=\frac{1}{2}$, $r=-1$.

In terms of the variables introduced in II., the problem and its associated Main Theorem are written

$$x: \quad x(t) \quad t^0 \leq t \leq t^1$$

$$[t^0, x(t^0)], \quad [t^1, x(t^1)] \quad \text{fixed}$$

$$J(x) = \int_{t^0}^{t^1} [t, x(t), \dot{x}(t)] dt = \min$$

$x_0: x_0(t), \quad t^0 \leq t \leq t^1$ is the minimizing arc.

Main Theorem: 1) $f - \dot{x}f_{\dot{x}}$ is continuous along x_0 and

$$\frac{d}{dt} (f - \dot{x}f_{\dot{x}}) = f_t \text{ on } x_0; \quad (3.6)$$

$f_{\dot{x}}$ is continuous along x_0 and

$$\frac{d}{dt} f_{\dot{x}} = f_x \text{ on } x_0 \text{ and} \quad (3.7)$$

$$2) \quad E(t, x_0(t), \dot{x}_0(t), \dot{X}) \geq 0 \quad (3.8)$$

for all $(t, x(t), \dot{x}(t))$ in R , where

$$E(t, x, \dot{x}, \dot{X}) = f(t, x, \dot{X}) - f(t, x, \dot{x}) - (\dot{X} - \dot{x})f_{\dot{x}}(t, x, \dot{x}).$$

It is easy to give a graphical interpretation of the Weierstrass condition. Let $Z = f(y')$, holding x and y fixed. In the z - y' plane, Fig. 7, at the point $y'_0, z_0 = f(y'_0)$, draw the indicatrix $z - z_0 = f_{y'}(y'_0)(Y' - y'_0)$, i.e., the tangent to the curve at that point. We see that $f(Y') \geq f(y'_0) + (Y' - y'_0)f_{y'}(y'_0)$. Thus the Weierstrass condition is interpreted as the condition that the curve $z = f(y')$ lies everywhere above the indicatrix in the neighborhood of the minimum y_0 .

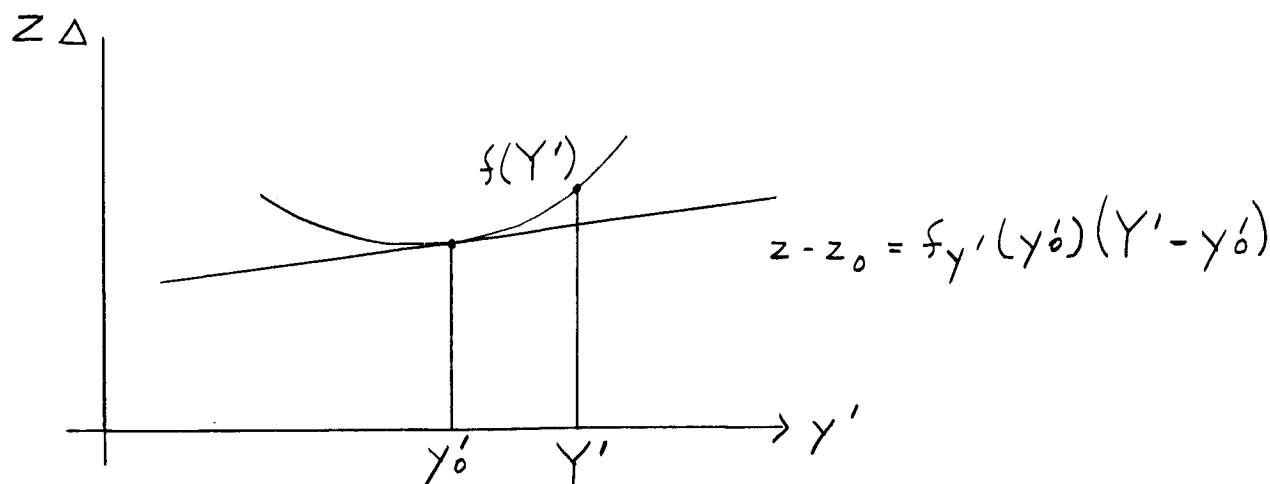


Fig. 7

Before giving the proof of the Main Theorem, we must make some qualitative definitions.

Weak Neighborhood: For a given interval $(x_1 \leq x \leq x_2)$, an arc $x, y(x), y'(x)$ is said to lie in a weak neighborhood of another arc $x, y_0(x), y'_0(x)$, if $y(x)$ and $y'(x)$ differ little from $y_0(x)$ and $y'_0(x)$ in the interval.

Strong Neighborhood: For a given interval $(x_1 \leq x \leq x_2)$, an arc $x, y(x), y'(x)$ is said to lie in a strong neighborhood of another arc $x, y_0(x), y'_0(x)$, if $y(x)$ differs little from $y_0(x)$. The Euler equations are derived using the concept of a weak neighborhood; the Weierstrass condition is based on the concept of a strong neighborhood.

Let us prove the Main Theorem in terms of variables used in the second statement of it, (3.6), (3.7) and (3.8).

Let the function

$$h: h(t), \quad t^0 \leq t \leq t^1$$

be an admissible (weak) variation, i.e., $h(t^0)=0$, $h(t^1)=0$, so that the function

$$x_0 + \epsilon h: x_0(t) + \epsilon h(t), \quad t^0 \leq t \leq t^1$$

has the same end points as $x_0(t)$. ϵ is an arbitrary, small number. We write

$$\begin{aligned} \phi(\epsilon) = J(x_0 + \epsilon h) &= \int_{t^0}^{t^1} f(t, x_0(t) + \epsilon h(t), \dot{x}_0(t) + \\ &\quad \epsilon \dot{h}(t)) dt. \end{aligned} \quad (3.9)$$

In order that x_0 be a minimizing arc,

$$\phi'(0) = 0 \quad \text{and} \quad \phi''(0) \geq 0.$$

Hence,

$$0 = \phi'(0) = J'(x_0, h) = \int_{t^0}^{t^1} (f_x h + f_{\dot{x}} \dot{h}) dt \quad \text{and} \quad (3.10)$$

$$0 = \phi''(0) = J''(x_0, h) = \int_{t^0}^{t^1} 2\omega(t, h, \dot{h}) dt, \quad (3.11)$$

$$\text{where } 2\omega = f_{xx} h h + 2f_{x\dot{x}} h \dot{h} + f_{\dot{x}\dot{x}} \dot{h} \dot{h}.$$

(3.10) is analogous to the directional derivative introduced in II. Let us rewrite (3.10) as

$$J'(x_0, h) = \int_{t^0}^{t^1} [M(t)h(t) + N(t)\dot{h}(t)] dt. \quad (3.12)$$

We now state a

Fundamental Lemma:

If $M(t)$ and $N(t)$ are piecewise continuous, then

$$\int_{t^0}^{t^1} (Mh + N\dot{h})dt = 0 \text{ for all admissible } h,$$

$$\text{if and only if } N(t) = \int_{t^0}^t M(s)ds + N(t^0).$$

Proof: Let $q(t) = \int_{t^0}^t M(s)ds$, i.e., $\dot{q}(t) = M(t)$, and put
 $h(t) = \int_{t^0}^t [N(r) - q(r)]dr - C(t-t^0)$, i.e., $h(t^0) = 0$. If
 we choose C such that $h(t^1) = 0$, then $h(t)$ is admissible.
 Let $p(r) = q(r) + C$, and write

$$h(t) = \int_{t^0}^t [N(r) - p(r)]dr.$$

$$\begin{aligned} \text{Then } \dot{p}(r) &= \dot{q}(r) = M(r), \\ \dot{h} &= N - p, \end{aligned}$$

and finally,

$$\int_{t^0}^{t^1} (M(t)h + N(t)\dot{h})dt = \int_{t^0}^{t^1} (\dot{p}h + p\dot{h})dt = p h \Big|_{t^0}^{t^1} = 0. \quad \text{Q.E.D.}$$

The proof of (3.6) and (3.7) of the Main Theorem follows directly from the Lemma.

To prove the Weierstrass condition (3.8), we refer to Fig. 8. We will admit strong variations of the form shown, calling the variation $X(t)$, $t_0 \leq t \leq t_0 + \epsilon$ ($\epsilon \geq 0$) and $X(t + \epsilon)$, $t_0 + \epsilon \leq t \leq t^1$. Note that $X(t, 0) = x_0(t)$.

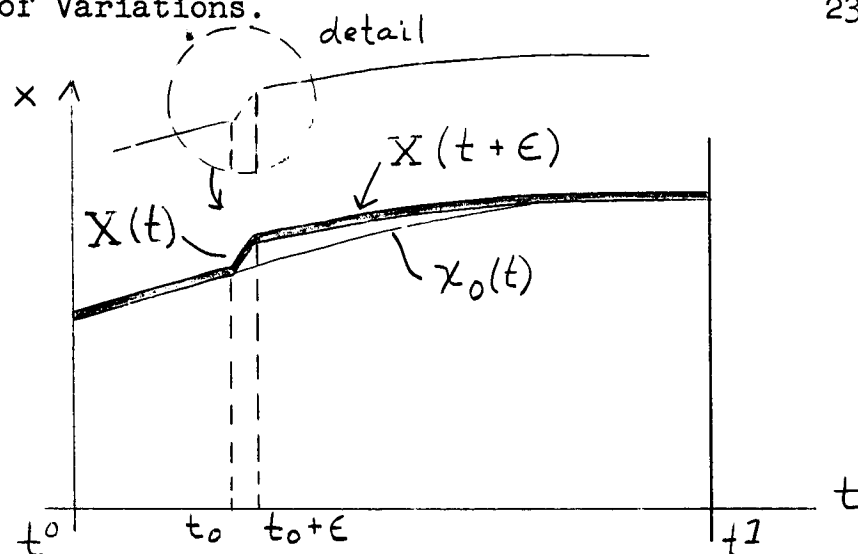


Fig. 8

For the arc with corners,

$$J(x_0) = \phi(0) \leq \phi(\epsilon) = \int_{t^0}^{t_0} f(t, x_0(t), \dot{x}_0(t)) dt + \\ \int_{t_0}^{t_0+\epsilon} f(t, X(t), \dot{X}(t)) dt + \int_{t_0+\epsilon}^{t^1} f(t, X(t+\epsilon), \dot{X}(t+\epsilon)) dt.$$

Hence,

$$0 \leq \dot{\phi}(0) = f(t_0, x_0(t_0), \dot{X}) - f(t, x_0(t), \dot{x}_0(t)) + \\ \int_{t_0}^{t^1} (f_x x_\epsilon + f_{\dot{x}} \dot{x}_\epsilon) dt.$$

By the Fundamental Lemma, the integral becomes $f_{\dot{x}} x_\epsilon \Big|_{t_0}^{t^1}$.

Note that $x_\epsilon(t^1, 0) = 0$ and $x_\epsilon(t_0, 0) = \dot{X} - \dot{x}_0(t)$.
Thus we have the Weierstrass condition

$$0 \leq \phi'(0) = f(t_0, x_0(t_0), \dot{x}) - f(t_0, x_0(t_0), \dot{x}_0(t_0)) - (\dot{x} - \dot{x}_0(t_0))f_{\dot{x}}(t, x_0, \dot{x}_0).$$

Transversality conditions arise in variational problems in which one or both end points are not fixed. For example, in finding the shortest distance between a point P in a plane and a curve $y_1(x)$ in the same plane, one end point is fixed at P and the other is variable. It is clear that the minimal curve y_0 will be the straight line which is normal to the curve and passes through P , Fig. 9. It will have the direction $(1, y'_1)$, and the end-point condition is

$$(1, y'_1) \perp dx, dy_1. \quad (3.13)$$

(3.13) is called the transversality (normality) condition.

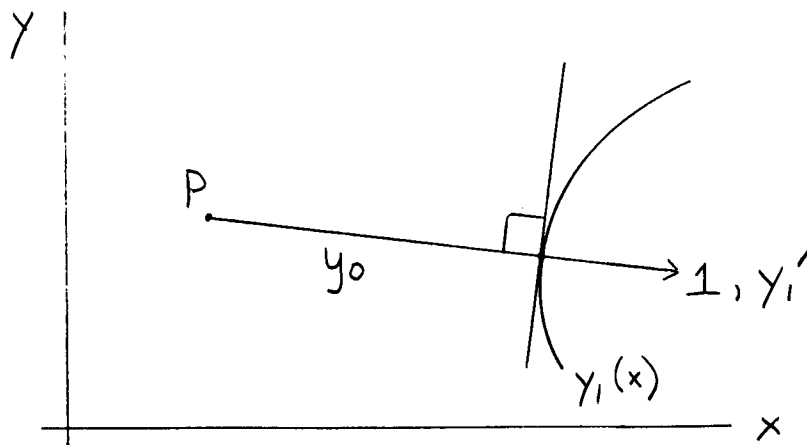


Fig. 9

For the general problem

$$J = \int_{x_0}^{x_1} f(x, y, y') dx = \min$$

with x_1 constrained to be on some curve $y_1(x)$, the transversality condition to be satisfied at the variable end point x_1 is that line with the directions

$f - y'_1 f_{y'_1}$, and $f_{y'_1}$ must be perpendicular to $dx dy_1$. Hence

$$(f - y'_1 f_{y'_1})dx + f_{y'_1} dy_1 = 0. \quad (3.14)$$

Let us prove (3.14) in complete generality in terms of the variables used in the proof of the Weierstrass condition.

$$x: x^1(t), \quad t^0 \leq t \leq T,$$

$$[t^0, x^1(t^0)] \text{ held fast,}$$

$T, x^1(T)$ are constrained to lie on a surface S .

The problem is

$$J(x) = g[T, x^1(T)] + \int_{t^0}^T f(t, x(t), \dot{x}(t)) dt = \min. \quad (3.15)$$

Let $x_0: x_0(t) \quad t^0 \leq t \leq T_0$ be the solution, and choose a one-parameter family of curves $x(t, \epsilon), t^0 \leq t \leq T(\epsilon)$ joining the initial point to a point $T(\epsilon), X(\epsilon) = x(T(\epsilon), \epsilon)$ on S , such that it contains the point x_0 for $\epsilon=0$, i.e., $x(t, 0) = x_0, T(0) = T_0$. We form the function

$$J(\epsilon) = g[T(\epsilon), X(\epsilon)] + \int_{t^0}^{T(\epsilon)} f(t, x(t, \epsilon), \dot{x}(t, \epsilon)) dt.$$

Then

$$dJ = dg + f(T)dT + \int_{t_0}^{T_0} (f_x \delta x + f_{\dot{x}} \delta \dot{x}) dt, \quad (3.16)$$

where $\delta x = x \delta \epsilon$, and we have put $\epsilon=0$, $d\epsilon \equiv 1$.

(3.16) must vanish if x_0 is the minimal arc. The Euler equations (3.6) and (3.7) must hold. Hence,

$$dJ = dg + f dT = 0.$$

Integrating $f dT$ by parts gives

$$dg + \left[(f - \dot{x}^1 f_{\dot{x}^1}) dT + f_{\dot{x}^1} dX^1 \right]_{t=T_0}^{t=T_0} = 0.$$

If g is absent from (3.15), then we have the transversality condition given above. If not, then the expression in square brackets must be equal to $-dg$.

Before we leave the classical theory we will discuss briefly the theory of multiple integrals. Consider

$$J = \iint_A f(x, y, z, p, q) dx dy, \quad (3.17)$$

where $p = \partial z / \partial x$ and $q = \partial z / \partial y$. The gradient of the functional J is

$$\delta J = \iint_A (f_z \delta z + f_p \delta z_x + f_q \delta z_y) dx dy. \quad (3.18)$$

If we define the inner product of the two functions u and v ,

$$(u, v) = \iint_A (u_x v_x + u_y v_y) dx dy,$$

then the gradient is $(u, \delta z)$. Suppose $\delta z=0$ on the boundary C of the region R' , the projection of the region R of definition $z(x,y)$, onto the xy plane.

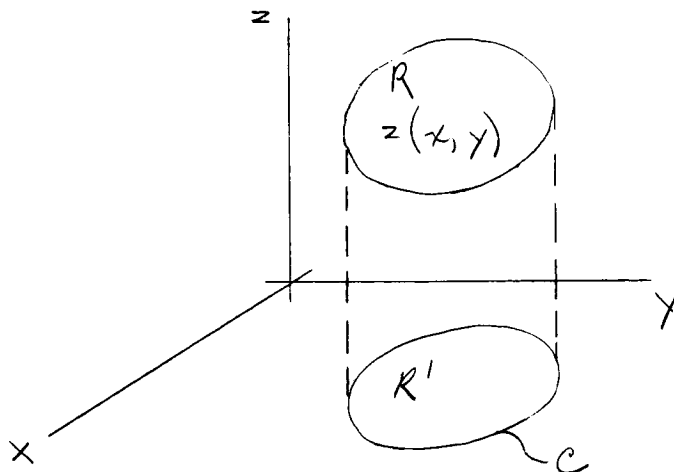


Fig. 10

Then from (3.18), the Euler equation is

$$\Delta u = \frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q - f = 0 \quad (3.19)$$

and $u=0$ on the boundary.

Finally, let us discuss briefly functionals containing higher derivatives:

$$x: \quad x(t) \quad t^0 \leq t \leq t^1$$

$$J(x) = \int_{t^0}^{t^1} f(t, x, \dot{x}, \ddot{x}) dt = \min,$$

and let $x(t^0)$, $\dot{x}(t^0)$, $x(t^1)$, $\dot{x}(t^1)$ be held fast. The Euler equation, which can again be derived by means of the directional derivative concept, is

$$f_x - \frac{d}{dt} f_{\dot{x}} + \frac{d^2}{dt^2} f_{\ddot{x}} = 0, \quad (3.20)$$

and the Weierstrass condition is as before with

$$E(t, x, \dot{x}, \ddot{x}, \ddot{X}) = f(t, x, \dot{x}, \ddot{X}) - f(t, x, \dot{x}, \ddot{x}) - (\ddot{X} - \ddot{x})f_{\ddot{x}}(t, x, \dot{x}, \ddot{x}).$$

It is interesting to note how the above problem can be cast into the form of a control problem, as introduced earlier or discussed in more detail in V. Write

$$x^1 = x, \quad x^2 = \dot{x}, \quad u = \ddot{x}.$$

Then the differential equations of the process are

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = u,$$

with $x^1(t^0)$, $x^1(t^1)$ given. Then we wish to find x, u for which

$$J = \int_{t^0}^{t^1} f(t, x^1, x^2, u) dt = \min,$$

which is a "control problem".

IV. Theory of Cones.

The theory of cones in n -dimensional geometry is useful for discussing advanced theories of the calculus of variations. The following is a brief introduction to the theory.

If we have a vector $k = (k_0, k_1, k_2, \dots, k_m)$, we define the following:

Def: A Hyperplane is the plane $L(k)=0$ where

$$\begin{aligned} L(k) &= \alpha_0 k_0 + \alpha_1 k_1 + \dots + \alpha_m k_m \\ \text{or } L(k) &= \lambda_0 k_0 - \lambda_1 k_1 - \dots - \lambda_m k_m \\ &= \lambda_0 - \lambda_\gamma k_\gamma \quad (\gamma=1, 2, \dots, m). \end{aligned}$$

For example, in two dimensions, a hyperplane is any line through the origin. A hyperplane divides the $m+1$ dimensional space into two half spaces $L(k) \geq 0$ and $L(k) \leq 0$.

Def: A Ray is a vector $k \neq 0$ and all $\alpha k (\alpha \geq 0)$, i.e., all non-negative multiples of a vector.

Def: A Cone K is a collection of rays. If k is in K , so also is $\alpha k, \alpha \geq 0$.

Def: A Convex Cone K is a cone such that if k and k' are in K , so also is $k+k'$, and $\alpha k + \alpha' k', \alpha \geq 0, \alpha' \geq 0$. It is necessary that $\sum \alpha \neq 1$.

Lemma I: If K and \bar{K} are convex cones such that no \bar{k} in \bar{K} is interior to K , there exists a hyperplane $L(k)=0$ which separates them into sets such that

$$\begin{aligned} L(k) &\geq 0 \text{ if } k \text{ is in } K \\ L(\bar{k}) &\leq 0 \text{ if } \bar{k} \text{ is in } \bar{K}. \end{aligned}$$

Def: A Tangent Cone to a region R at a point x_0 is the cone defined by the limiting positions of the rays through x_0 , directed into R , as they approach the positions for which they are no longer in or on the boundary of R .

For example, for a smooth closed region R , the cone tangent to R at a point x_0 on the boundary is the half space containing R . If x_0 is interior to R , the tangent cone is the whole space. At a boundary point of a region R with corners, the tangent cone may be less or more than a half space, depending on whether the corner is re-entrant.

Theorem: Let X be a closed, well-behaved set in x -space, $x = (x^1, x^2, \dots, x^m)$, let x_0 be a boundary point of X , and let \mathcal{C} be the cone tangent to X at x_0 . Assume that \mathcal{C} is convex and has an interior point. Let $f_0(x)$, $f_1(x)$, ..., $f_m(x)$ be functions on X having derivatives $f'_0(x_0, h)$, $f'_1(x_0, h)$, ..., $f'_m(x_0, h)$ at x_0 , and let K be all vectors k defined by the formula

$$k_\rho = f'_\rho(x_0, h), \text{ where } h \text{ is in } \mathcal{C}, (\rho=0,1,\dots,m).$$

Then K is a closed convex cone.

A proof of this Theorem will not be given here.

Lemma II: If \bar{k} is interior to K , there is an \bar{h} interior to \mathcal{C} such that $\bar{k}_\rho = f'_\rho(x_0, \bar{h})$, and there exists a curve $x(t)$ such that

$$x^1(t) = x^1_0(t)$$

and

$$f_\rho(x(t)) - f_\rho(x_0) = t \bar{k}_\rho \quad (0 \leq t \leq \epsilon), \text{ i.e.,}$$

$$\text{and} \quad \dot{x}(0) = h.$$

Let us now apply some of these results to one of the problems we considered earlier. Suppose that x_0 is a

solution of

$$f_0(x) = \min$$

$$\text{subject to } f_\alpha(x) = 0 \quad (\alpha=1,2,\dots,m')$$

$$\text{and } f_\beta(x) \geq 0 \quad (\beta=m'+1,\dots,m),$$

which means

$$f_\alpha(x_0) = 0$$

$$f_{\beta'}(x_0) = 0 \quad \beta'=m'+1,\dots,m''$$

$$f_{\beta''}(x_0) > 0 \quad \beta''=m''+1,\dots,m,$$

and let \bar{K} be all $\bar{k} = (\bar{k}_0, \dots, \bar{k}_m)$, i.e., $\bar{k}_0 < 0$, $\bar{k}_\alpha = 0$, $\bar{k}_{\beta'} \geq 0$, $\bar{k}_{\beta''}$ arbitrary. Then no \bar{k} in \bar{K} is interior to K , where K is defined by the previous Theorem.

To see these results, we suppose the last statement is untrue and show a contradiction. If it is untrue, then by Lemma II,

$$f_0(x(t)) = f_0(x_0) + t\bar{k}_0 \quad 0 \leq t \leq \epsilon$$

$$f_\alpha(x(t)) = f_\alpha(x_0) + t\bar{k}_\alpha$$

$$f_{\beta'}(x(t)) = f_{\beta'}(x_0) + t\bar{k}_{\beta'}, \quad \text{and}$$

$$f_{\beta''}(x(t)) = f_{\beta''}(x_0) + t\bar{k}_{\beta''}.$$

But the first equation leads to the conclusion that $f_0(x(t)) > f_0(x_0)$, because $t > 0$, $k_0 < 0$, a clear contradiction.

Theorem: If x_0 is a solution of the above problem, there exist multipliers $\lambda_0 \geq 0$, $\lambda_1, \dots, \lambda_m$ such that

$$1) \quad \lambda_{\beta'} \geq 0$$

$$2) \quad \lambda_{\beta''} = 0$$

- 3) The function $F = \lambda_0 f_0 - \lambda_\gamma f_\gamma$ has the property that $F'(x_0, h) \geq 0$ for all h in \mathcal{C} .

Proof of 3):
$$\begin{aligned} F'(x_0, k) &= \lambda_0 f'_0(x_0, h) - \lambda_\gamma f'_\gamma(x_0, h) \\ &= \lambda_0 k_0 - \lambda_\gamma k_\gamma \\ &= L(k) \geq 0 \text{ for } k \text{ in } K. \end{aligned}$$

Proof of 1) and 2):

Choose $\bar{k} = (-1, 0, 0, \dots, 0)$ in \bar{K} . Then $L(\bar{k}) = -\lambda_0 \leq 0 \therefore \lambda_0 \geq 0$.

Now choose $\bar{k} = (-1, 0, \dots, 0, \bar{k}_\sigma, 0, \dots, 0)$ such that there are at least $m'+1$ zeros before \bar{k}_σ and at least $m+1-m''$ zeros after \bar{k}_σ . Then

$$L(\bar{k}) = -\lambda_0 - \lambda_\sigma \bar{k}_\sigma \leq 0, \quad m'+1 \leq \sigma \leq m'',$$

where if \bar{k}_σ is any positive number, $\lambda_\sigma > 0$, and if \bar{k}_σ is any non-positive number, $\lambda_\sigma = 0$.

V. Control Theory.

In control problems it is customary to think of the states of the systems being controlled as being represented by the vector

$$x: x(t) = (x^1(t), x^2(t), \dots, x^q(t))$$

and the control by another vector

$$u: u(t) = (u^1(t), u^2(t), \dots, u^n(t)).$$

The process, as it takes place in time, is governed by differential equations

$$\dot{x}^1 = f^1(t, x, u),$$

and usually starts at some initial point

$$[t^0, x^1(t^0)] = b^1.$$

A given choice of $u(t)$ gives an initial value problem for the state

$$\dot{x}^1 = f^1(t, x(t), u(t)) = g^1(t, x)$$

$$[t^0, x^1(t^0)] = b^1.$$

However, the problem in control theory is to determine $u(t)$ such that we hit some target while minimizing something, say time, fuel consumption or money.

An example of a simple control problem is to choose $u(t)$ such that at a fixed time T you reach $x^1(T) = c^1$ in such a way that

$$J = \int_{t^0}^T (f(t), x(t), u(t)) dt = \min. \quad (5.1)$$

It can be seen that this problem is contained in the classical variational problem discussed in I., when T is replaced by t^1 and $u^1(t)$ is replaced by $\dot{x}^1(t)$.

The problem can be modified in several ways to make it more meaningful, but more complicated. We could add

constraints of the form

$$|u^1(t)| \leq c$$

or, say inequality constraints

$$\begin{aligned} \phi \alpha(t, u(t)) &\geq 0 \\ \phi \beta(t, u(t)) &= 0 \quad \text{or} \\ \phi \alpha(t, x(t), u(t)) &\geq 0, \text{ etc.} \end{aligned}$$

Let us translate the above problems into the language and notation of Pontryagin [1]. Let

$$p_1(t) = f_{x_1}(t, x_0(t), \dot{x}_0(t)) \quad (5.2)$$

and let

$$u^1 = \dot{x}^1, \quad u_0^1(t) = \dot{x}_0^1(t).$$

We now define a new function

$$H(t, x, u, p) = p_1 f_1 - f(t, x, u). \quad (5.3)$$

The minimizing arc u_0, x_0 has the property that

$$H(t, x_0(t), u, p(t)) \leq H(t, x_0(t), u_0(t), p(t)), \quad (5.4)$$

i.e., H is minimized over all admissible functions u .

Hence

$$H_{u^1} = p_1 - f_{\dot{x}^1}(t, x, u).$$

The classical Weierstrass condition comes directly from (5.4):

$$\begin{aligned}
0 &\leq H(t, x_0, u_0, p) - H(t, x_0, u, p) \\
&= p_1(t)u_0^1 - f(t, x_0, u_0) - [p_1 u^1 - f(t, x_0, u)] \\
&= f(t, x_0, u) - f(t, x_0, u_0) - (u^1 - u_0^1) f_{\dot{x}^1}(t, x_0, u_0) \\
&= E(t, x_0, u_0, u).
\end{aligned}$$

At this point one can make an analogy to the theory of Hamilton-Jacobi dynamics. If H were the Hamiltonian, then the Hamilton-Jacobi equations would be

$$\begin{aligned}
\dot{x}_1 &= H_{p_1} = u^1 \\
\dot{p}_1 &= -H_{x^1} = f_{x^1}.
\end{aligned} \tag{5.5}$$

The Hamiltonian would be defined by the definition of $H(t, x, u, p)$ and the equation

$$H_{u^1} = 0.$$

In the classical variations theory, (5.5) are the Euler equations.

Recall the modern brachistochrone problem dealt with earlier:

$$\dot{x}^1 = x^3, \dot{x}^2 = x^4, \dot{x}^3 = F \cos u, \dot{x}^4 = F \sin u - g$$

with $[x^1(0), x^1(T)]$ given, choose u such that $T = \min$.

This problem also fits very easily into the general context of Leitman [2], who discusses problems of the form

$$\ddot{x} = X(t, x, y) + \frac{c\beta}{m} \cos \psi$$

$$\ddot{y} = Y(t, x, y) + \frac{c\beta}{m} \sin \psi$$

$$\dot{m} = -\beta \quad 0 \leq \beta \leq \beta_{\max}$$

$$G(T, x(T), y(T), \dot{x}(T), \dot{y}(T), m(T)) = \min,$$

with an initial point given. Such a problem is called a Problem of Mayer in classical texts.

Let us now state the necessary conditions for the solution to the following general control problem:

x : state variable $x^i(t)$ ($i=1, \dots, q$)
 u : control variable $u^k(t)$ ($k=1, \dots, u$),

where $t^0 \leq t \leq T$.

The governing differential equations are

$$\dot{x}^i = f^i(t, x, u).$$

We are given $[t^0, x^1(t^0)]$ fixed and $x^1(T)$ fixed, and we wish to make

$$J(x) = g(T) + \int_{t^0}^T f(t, x, u) ds = \min. \quad (5.6)$$

Assume that $x_0^1(t), u_0^k(t), t^0 \leq t \leq T_0$ is the solution, and define as before the function

$$H(t, x, u, p) = p_1 \dot{x} - \lambda_0 f. \quad (5.7)$$

Then there exist multipliers $\lambda_0 \geq 0$ and $p_1(t)$, not all zero, such that

$$\begin{aligned} \dot{x} &= H_{p_1} = f \\ \dot{p}_1 &= -H_{x_1}, \end{aligned} \quad (5.8)$$

the Euler equations, and

$$H(t, x_0(t), u, p(t)) \leq H(t, x_0(t), u_0(t), p(t)) \quad (5.9)$$

for all $(t, x_0(t), u)$ that are admissible. Admissibility may be defined by constraints of the following general form:

$$0 \leq u^k \leq c, \quad |u^k| \leq c, \quad \phi_\alpha(u) \geq 0.$$

Equation (5.9) is the Weierstrass condition for this problem.

The transversality condition takes the form

$$\lambda_0 g'(T) - H(T, x_0(T), u_0(T), p(T)) = 0, \quad (5.10)$$

The analogous form of the transversality condition for the classical approach is given in III.

Let us solve the rocket problem (modern brachistochrone):

$$H = p_1 \dot{x}^3 + p_2 \dot{x}^4 + F(p_3 \cos u + p_4 \sin u) - p_4 g.$$

But $\frac{d}{dt} H = H_t = 0$; therefore $H = \text{constant} = \lambda_0 g'(T)$;
 $g(T) = T$; hence $H = \lambda_0 \geq 0$ along the minimal curve. Now

$$\dot{p}_1 = -H_{x^1} = 0 \quad \therefore p_1 = \text{constant}$$

$$\dot{p}_2 = -H_{x^2} = 0 \quad \therefore p_2 = \text{constant}$$

$$\dot{p}_3 = -H_{x^3} = -p_1 \quad \therefore p_3 \text{ is linear in } t$$

$$\dot{p}_4 = -H_{x^4} = -p_2 \quad \therefore p_4 \text{ is linear in } t.$$

Let $\xi = p_3$, $\eta = p_4$, and we see that $\dot{\xi} = 0$, and $\dot{\eta} = 0$; hence the point (ξ, η) moves at a constant rate. Since we have no constraints of the form $\phi_\alpha(u) \geq 0$, we must choose u such that $H = \max$ on the minimal curve, $H_u = 0$ on the curve.

$$0 = H_u = F(-p_3 \sin u + p_4 \cos u) = 0.$$

$$\text{Hence } \tan u = \frac{p_4}{p_3} = \frac{\eta}{\xi}.$$

The properties of the solution have been obtained without finding an explicit solution. The solution says that the thrust force F is always directed to a point that moves on a straight line at a constant rate, Fig. 11.

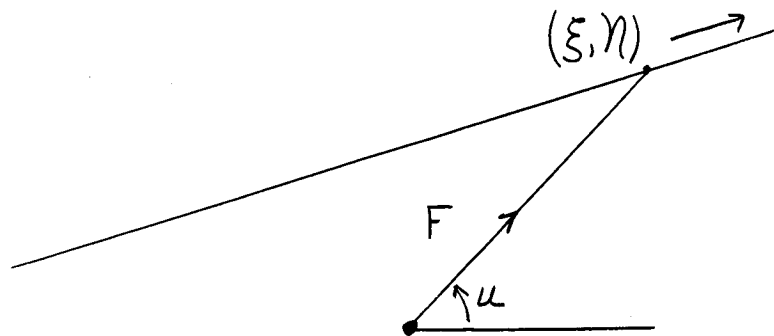


Fig. 11

Not all problems in control theory have solutions, i.e., not all systems are controllable. To illustrate the concept of controllability, let us suppose a problem is governed by a set of differential equations

$$\dot{x}^1 = f^1(t, x, u).$$

We now ask whether there are functions u which can get us from P_0 to P_1 .



If we can get to P_1 , can we get to P_2 , a neighboring point, also? It might not be possible. To be explicit, consider the geodesic problem

$$y: y(x) \quad x_1 \leq x \leq x_2$$

$$J(y) = \int_{x_1}^{x_2} \sqrt{1+y'^2} \, dx = \min.$$

Let us introduce the function

$$z(x) = \int_{x_0}^x \sqrt{1+y'^2} \, dx,$$

and put $y' = \tan u$, $z' = \sec u$, $x_1 = 0$, $x_2 = 1$. The problem is now

$$y(x_1) = 0$$

$$z(x_1) = 0$$

$$z(x_2) = \int_0^1 \sqrt{1+y'^2} \, dx = \min,$$

which is a Problem of Mayer. The solution $n=\text{constant}$ is known a priori. Let us assume that $y(x_2)=b$ and $x_2=1$. The properties of the solution can be most easily shown in a figure.

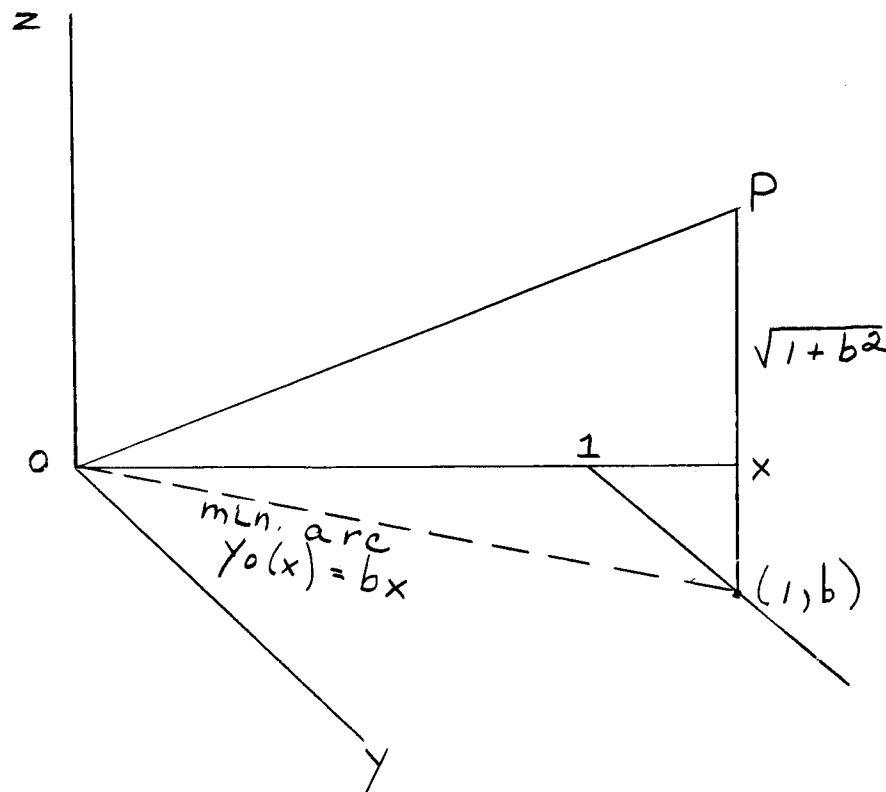
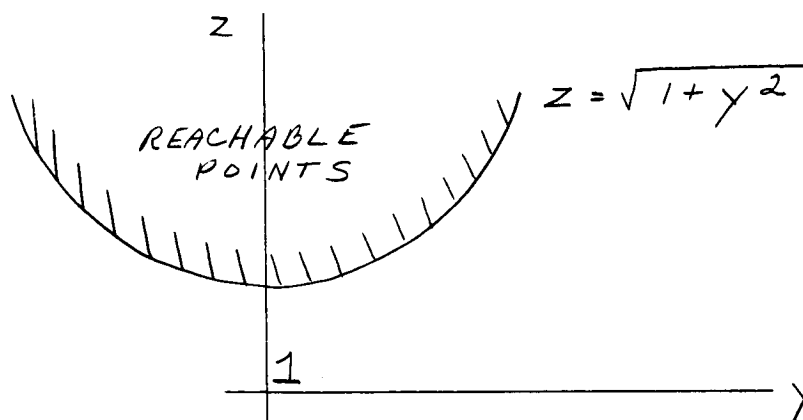


Fig. 12

The line OP represents the locus of points of Z for the solution curve $y_0(x)$. For any variation from the true solution, the corresponding value of Z must be larger than the value of Z on OP for the same set of values of (x,y) . We see that there is a hyperbolic cone of reachable points. The line OP is on the boundary of the cone. If we draw the intersection of the cone with the plane $x_2=1$, we obtain Fig. 13. Even for the simple problem discussed here, there may be points $z(x_2)$ that cannot be reached, regardless of the control available.



This lack of complete controllability is typical of Problems of Mayer.

Finally, let us discuss a particular case of the control problem where we have constraints of the form

$$|u| \leq 1.$$

Let the problem be to approach the origin in \dot{x}, x phase space in minimum time, subject to a control u and differential equations

$$\begin{aligned}\dot{x}^1 &= x^2 \\ \dot{x}^2 &= u.\end{aligned}$$

The function $H = p_1 x^2 + p_2 u$. We must choose u to maximize H subject to $|u| \leq 1$. Carrying out the steps

$$\begin{aligned}\dot{p}_1 &= 0 & \text{hence } p_1 &= c_1 = \text{constant} \\ \dot{p}_2 &= -p_1 & \text{hence } p_2 &\text{ is linear in } t,\end{aligned}$$

i.e., $p_2 = c_2 - c_1 t$. For fixed time, we will maximize H by selecting u . It is straightforward to show that if $p_2(t) > 0$, $u=1$, and if $p_2(t) < 0$, $u=-1$. The solution can be written

$$u = \text{sign}(c_2 - c_1 t)$$

in phase space.

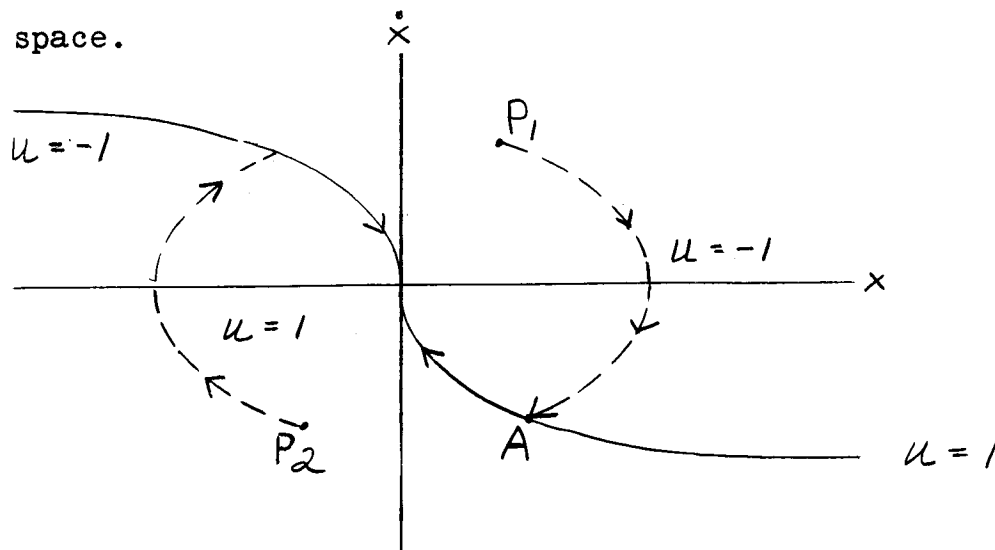


Fig. 13

Starting at, say point P_1 in phase space, the trajectory follows the curve shown. First $u=-1$ up to point A ; then $u=1$ to the origin. Similar remarks hold for point P_2 .

VI. Problem Transformations.

By means of simple transformations, we can show that all of the above problems are, in principle, the same.

The general control problem is given in terms of

$$\begin{array}{lll} x: & x^i(t) \text{ state variables} & t^0 \leq t \leq t^1 \quad i=1, \dots, q \\ u: & u^k(t) \text{ control variables} & k=1, \dots, n, \end{array}$$

subject to differential equations

$$\dot{x}^1 = f^1(t, x, u),$$

and the formulation may depend on other parameters

$$w^\sigma \quad \sigma = 1, \dots, r,$$

and may be constrained by conditions such as

$$\begin{aligned} \phi_\alpha(u) &= 0 \\ \phi_\beta(u) &\geq 0 \\ \phi_\alpha(t, u) &= 0 \\ \phi_\alpha(t, x, u) &\geq 0, \text{ etc.} \end{aligned}$$

As Case (i), consider the constraint

$$\phi_\alpha(t, x, u) = 0 \quad (6.1)$$

with end points expressed parametrically as

$$\begin{aligned} t^0 &= T^0(w), & x^1(t^0) &= X^{10}(w) \\ t^1 &= T^1(w), & x^1(t^1) &= X^{11}(w). \end{aligned} \quad (6.2)$$

We impose isoperimetric conditions

$$J_1(x) = g(w) + \int_{t^0}^{t^1} f(t, x(t), u(t)) dt = 0, \quad (6.3)$$

and we wish to make

$$J(x) = g(w) + \int_{t^0}^{t^1} f(t, x(t), u(t)) dt = \min. \quad (6.4)$$

If we have a problem with constraints of the form

$$\begin{aligned}\phi_{\alpha'}(t, x, u) &= 0 & \alpha' &= 1, \dots, m' \\ \phi_{\alpha''}(t, x, u) &\geq 0 & \alpha'' &= m'+1, \dots, m,\end{aligned}$$

then to get the formulation, Case (i), we can introduce more functions u by writing

$$\begin{aligned}\phi_{m'+j}(t, x, u) &= (u^{n+j})^2 \\ \text{or} \\ \phi_{m'+j}(t, x, u) - (u^{n+j})^2 &= 0 & j &= 1, \dots, m-m',\end{aligned}$$

which are just $m-m'$ more constraints of the desired form $\phi(t, x, u) = 0$. This method, however, does introduce singularities, so caution is in order. Isoperimetric inequalities can be similarly transformed. In principle, Case (i) contains all problems which include inequality constraints.

Let us discuss now the Isoperimetric Problem of Bolza:

$$x: x^i(t) \quad t^0 \leq t \leq t^1 \quad i=1, \dots, q$$

$$\text{with constraints } \phi_{\alpha}(t, x, \dot{x}) = 0 \quad \alpha=1, \dots, m$$

$$\begin{aligned}J_{\gamma}(x) &= g_{\gamma}(t^0, x(t^0), t^1, x(t^1)) + \\ &\quad \int_{t^0}^{t^1} f_{\gamma}(t, x(t), \dot{x}(t)) dt = 0 & \gamma=1, \dots, p.\end{aligned} \tag{6.5}$$

$$\begin{aligned}J_0(x) &= g_0(t^0, x(t^0), t^1, x(t^1)) + \\ &\quad \int_{t^0}^{t^1} f_0(t, x(t), \dot{x}(t)) dt = \min.\end{aligned} \tag{6.6}$$

Remarks similar to those made above hold here for inequality constraints.

Consider, as special cases

$$\text{Case (i)} \quad f_{\gamma} \equiv 0 \quad \text{the Problem of Bolza} \quad (6.7)$$

$$\text{Case (ii)} \quad \begin{aligned} f_{\gamma} &\equiv 0 \\ f_0 &\equiv 0 \end{aligned} \quad \text{the Problem of Mayer} \quad (6.8)$$

$$\text{Case (iii)} \quad \begin{aligned} f_{\gamma} &\equiv 0 \\ g &\equiv 0 \end{aligned} \quad \text{the Problem of Lagrange.} \quad (6.9)$$

We will show that all three problems are basically the same, first showing that the functions f can be eliminated, i.e., we can write equivalent problems involving no integrals.

Let the problem be

$$\begin{aligned} x: & x^1(t) \\ y: & y^{\rho}(t) \quad \text{where} \quad y^{\rho}(t) = \int_t^t \rho^{\rho} dt \quad \rho=0,1,\dots,p. \end{aligned}$$

The differential equations are now

$$\begin{aligned} \phi_{\alpha}(t, x, \dot{x}) &= 0 \\ \dot{y}^{\rho} - f_{\rho}(t, x, \dot{x}) &= 0, \end{aligned}$$

with side conditions

$$\begin{aligned} J_{\gamma}(x) = g_{\gamma} + y^{\gamma}(t) &= 0 \\ y^{\rho}(t^0) &= 0. \end{aligned}$$

The problem reduces to a Problem of Mayer, for we now wish

to make

$$J_0 = g_0 + y^0(t^1) = \min.$$

This transformation does not preserve the concept of strong neighborhoods.

Let us consider a more general Isoperimetric Problem of Bolza. Let the state variables be

$$x: x^i(t), w^\sigma \quad \begin{array}{l} i = 1, \dots, q \\ \sigma = 1, \dots, p \\ t^0 \leq t \leq t^1, \end{array}$$

assuming that the state also depends on parameters w , and with the differential equations

$$P_\alpha(t, x, \dot{x}) = 0. \quad (6.10)$$

We have end conditions

$$t^s = T^s(w), x^i(t^s) = X^{is}(w), s = 0, 1 \quad (6.11)$$

and constraints

$$J_\gamma(x) = g_\gamma(w) + \int_{t^0}^{t^1} f_\gamma(t, x, \dot{x}) dt = 0 \quad \gamma = 1, \dots, r, \quad (6.12)$$

and we wish to make

$$J(x) = g(w) + \int_{t^0}^{t^1} f(t, x, \dot{x}) dt = \min. \quad (6.13)$$

If w appears in the integrand of (6.12), we merely add the

new state variables $x^{q+\sigma}$ and the differential equations $\dot{x}^{q+\sigma} = 0$. Then the integrand in the constraints corresponding to (6.12) contain no terms in w^σ .

To see that the problem consisting of (6.5), (6.6) and (6.7) is a special case of this, let

$$t^0 = w^1, \quad x^1(t^0) = w^{1+1}, \quad t^1 = w^{q+2}, \quad x^1(t^1) = w^{q+2+1}.$$

The problem (6.10), (6.11), (6.12) and (6.13) is, conversely, identical to (6.5), (6.6) and (6.7). If we append to the set of differential equations associated with the latter problem the following,

$$\dot{w}^\sigma = 0, \quad \text{i.e.,} \quad w^\sigma = \text{constants, the end values,}$$

then we obtain the problem

$$x: \quad x^1(t), \quad w^\sigma(t).$$

$$\begin{aligned} \text{Differential equations} \quad & \dot{w}^\sigma = 0 \\ & \phi_\alpha = (t, x, \dot{x}), \end{aligned}$$

with end conditions becoming the constraints with $f_\gamma \equiv 0$,

$$t^0 - T^0(w(t)) = 0, \quad x^1(t^0) - X^{10}(w(t^0)) = 0,$$

$$t^1 - T^1(w(t^1)) = 0, \quad x^1(t^1) - X^{11}(w(t^1)) = 0,$$

and we wish to make

$$J = g(w(t^0)) + \int_{t^0}^{t^1} f(t, x, \dot{x}) dt = \min.$$

By similar arguments and transformations, it is possible to eliminate the constraint functions $J_\gamma(x)$ by transforming the variational problems to control problems. It is straightforward, conversely, to show that the control problem is a variational problem of one of the special types (6.7), (6.8) or (6.9). Thus, all of the special types of problems we have formulated and discussed are basically the same. The type of formulation one chooses is a matter of taste.

VII. Methods of Computation.

The method of steepest descent, or gradient method, can be most easily discussed in terms of functions of a finite number of variables. Let $f(x)$ be a function of n variables $x = (x^1, x^2, \dots, x^n)$. The derivative of f in the direction f is

$$f'(x, h) = g \cdot h = |g| \cdot |h| \cos \theta, \quad (7.1)$$

where $g = \text{grad } f = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n} \right)$, and θ is the angle between g and h . For fixed $|h|$, f' is greatest in the g direction.

Recall that at the minimum point x_0 we can expand

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0, x-x_0) + \dots \quad (7.2)$$

If we truncate this expression at the second order term and set $f(x) = \text{constant}$, we have an equation for an ellipsoid in x^n -space. Thus, starting with some approximate value of the solution x_0^a , we use the concept of the

gradient, or direction of greatest change of f , to follow the "flow lines" from some ellipsoid $f(x_0^a) = \text{constant}$ to the minimum point of f , i.e., we must solve the set of equations

$$\frac{dx^1}{dt} = -f_{x^1} = -g^1. \quad (7.3)$$

For numerical computations, (7.3) is discretized to

$$\Delta x^1 = -g^1 \Delta t$$

or the iterative form

$$x_{n+1}^1 = x_n^1 - \alpha g_n^1. \quad (7.4)$$

Equation (7.4) is the Gradient Iteration Formula, and embodies in it the Method of Steepest Descent.

An advantage of gradient methods is that they pull the solution away from saddle points. Usually the greatest difficulty in application is that one encounters long, narrow ellipsoids. To overcome this difficulty, one must apply special methods to choose α in (7.4).

To discuss the gradient method for integrals, consider the problem

$$x: x(t) \quad (t^0 \leq t \leq t^1)$$

$$[t^0, x(t^0)], [t^1, x(t^1)] \text{ held fast}$$

$$J(x) = \int_{t^0}^{t^1} (f(t), x(t), \dot{x}(t)) dt = \min.$$

We will admit corners in the minimizing arc

$$x_0: x_0(t) \quad (t^0 \leq t \leq t^1),$$

and we will call a variation h admissible if

$$h: h(t) \quad (t^0 \leq t \leq t^1)$$

and $h(t^0) = 0$; $h(t^1) = 0$. Note the vectorial character of $h(t)$. If h is admissible, so is αh . If, in addition, g is admissible, so is

$$\alpha_1 h + \alpha_2 g.$$

We define the inner product of h and g as

$$g \cdot h = (g, h) = \int_{t^0}^{t^1} \dot{g}(t) \dot{h}(t) dt. \quad (7.5)$$

Let us define

$$g(t) = \int_{t^0}^t \left[f_{\dot{x}}(\tau, x_0(\tau), \dot{x}_0(\tau)) - \int_{t^0}^{\tau} f_x(s, x_0(s), \dot{x}_0(s)) ds - c \right] d\tau, \quad (7.6)$$

where c is chosen so that

$$g(t^1) = 0. \quad (7.7)$$

Since $g(t^0) = 0$, g is an admissible variation. In fact, g is the gradient of J at x_0 , hence

$$J'(x_0, h) = \int_{t_0}^{t^1} \dot{g}(t) h(t) dt = g \cdot h, \quad (7.8)$$

$$\text{where } \dot{g}(t) = f_{\dot{x}} - \int_{t_0}^t f_{xx} ds - c.$$

Equation (7.8) must hold where

$$|h| = \int_{t_0}^{t^1} h^2 dt$$

is held fast. Note that $J'(x_0, h)$ has a maximum value for fixed $|h|$ when $h = \alpha g$.

For numerical solution of this type of problem, we can use the concepts developed earlier for functions of n variables, i.e., (7.4), but to use our definition of g , we would rewrite (7.4) as

$$x_{n+1}(t) = x_n(t) - \alpha g_n(t). \quad (7.9)$$

For a general discussion of gradient methods, see Stein, M., Jnl. of Research, Nat. Bur. Stds., RP2330, V.48, 1952.

The freedom of choice of the definition of gradient in these numerical methods is unconstrained. Suppose we define the dot or inner product of the functions $g(t)$, $h(t)$ as

$$g \cdot h = (g, h) = \int_{t_0}^{t^1} g(t) h(t) dt. \quad (7.10)$$

For J , defined as before,

$$J'(x_0, h) = \int_{t^0}^{t^1} (f_x h + f_{\dot{x}} \dot{h}) dt. \quad (7.11)$$

Integrating by parts with h taken as admissible, we obtain

$$J'(x_0, h) = \int_{t^0}^{t^1} g h dt \quad (7.12)$$

with

$$g = f_x - \frac{d}{dt} (f_{\dot{x}}). \quad (7.13)$$

We could call g in (7.13) the gradient. Analogous to (7.3), we would have to solve

$$\frac{\partial x}{\partial s}(t, s) = \frac{d}{dt} (f_{\dot{x}}) - f_x = -g \quad (7.14)$$

$x(t^0, s) = 0, \quad x(t^1, s) = 0$, where $x(t, 0) = x_0$.

For example, if

$$f = \frac{1}{2} \dot{x}^2 = \frac{1}{2} \left(\frac{\partial x}{\partial t} \right)^2$$

$$f_{\dot{x}} = \dot{x},$$

and we have the system

$$\frac{\partial x}{\partial s} = \frac{\partial^2 x}{\partial t^2},$$

with $x(t^0, s) = 0, x(t^1, s) = 0$, and $x(t, 0) = x_0(t)$. Note that by using the gradient approach in this simple example, we obtain a heat equation which gives the set of flow lines of the energy integral.

Of course, the above problem of minimization could have been handled by what Courant and Hilbert [3] call Indirect Methods, that is, by solving the corresponding

Euler equation

$$\frac{d}{dt} (f_{\dot{x}}) = f_x$$

subject to the two point conditions

$$x(t^0); \quad x(t^1) \quad \text{fixed.}$$

In the same book, Direct Methods are discussed. For example, if we define $\mu = \inf J(x)$ for all admissible x , the problem is to find μ by constructing a minimizing sequence x_q such that

$$\lim_{q \rightarrow \infty} J(x_q) = \mu.$$

On the other hand, we could approach the problem by a) finding μ , b) showing that $\bar{x}_q \rightarrow x_0$, and then c)

$\mu = \lim J(x_q) \geq J(x_0)$. The latter approach is that of the Tonelli School in Italy, and stems from work by Weierstrass and Hilbert.

As the first direct method, consider the following basically Eulerian technique for obtaining a minimizing sequence. Suppose the interval of interest is $(t^0 \leq t \leq t^1)$. Divide the interval into q sub-intervals of length

$$h = \frac{t^1 - t^0}{q},$$

where q is some integer. Then the integral $\int_{t^0}^{t^1} f dt$ can be written as a function of q variables $\xi_1, \xi_2, \dots, \xi_q$:

$$F(\xi_1, \xi_2, \dots, \xi_q),$$

and the minimum of this function can be obtained by the usual methods for functions of n variables. One disadvantage of this method is that it usually involves too many variables.

A second direct method, which is very useful when the side conditions are linear and the integrand functions are quadratic, is the Rayleigh-Ritz. The details of this method are discussed in Courant-Hilbert.

For another approach, we observe that the admissible variation

$$h = x(t) - x_0(t)$$

with the properties

$$h(t^0) = h(t^1) = 0.$$

We can estimate h by choosing a complete set of functions $h_k(t)$, $k=1,2,\dots$, which vanish at t^0 and t^1 , for example, if $t^0=0$:

$$h_k(t) = \sin \frac{\omega_k t}{t^1} \quad (\omega_k = k\pi). \quad (7.15)$$

We then write

$$h(t) = \sum_{k=1}^{\infty} \alpha_k h_k(t). \quad (7.16)$$

Hence

$$x(t) = x_0(t) + \alpha_1 h_1(t) + \dots + \alpha_q h_q(t) + \dots \quad (7.17)$$

Thus

$$J(x) = F(\alpha_1, \dots, \alpha_q) \quad (7.18)$$

if we terminate (7.17) at the q^{th} term. Thus we again have the problem of minimizing a function of q variables. The effectiveness of this method depends, as does the effectiveness of Rayleigh-Ritz, on the choice of the functions $h_k(t)$.

Side conditions of the form

$$K(x) = \int_{t_0}^{t_1} g(x, t) dt = C$$

merely impose on the resulting problem of minimizing $F(\alpha)$ conditions of the form

$$G(\alpha_1, \alpha_2, \dots, \alpha_q) = \text{constant}.$$

Iterative methods play a dominant role in the problem of minimizing integrals and functions of a finite number of variables. In general, we are given the task of finding the minimum of $F(x, y, z)$. If we guess a set x_0, y_0, z_0 that is close to the answer, we will get convergence of an iterative scheme, which we can approach as follows:

- | | | |
|----|---|-----------------|
| 1. | Given x_0, y_0, z_0 , minimize $F(x, y_0, z_0)$ | solution: x_1 |
| 2. | minimize $F(x_1, y, z_0)$ | solution: y_1 |
| 3. | minimize $F(x_1, y_1, z)$ | solution: z_1 |

and so on in this Gauss-Seidel-like procedure.

Let us define formally an iterative procedure. Let x_0 be an initial guess (a vector). Then we write

$$x_{q+1} = x_q + \alpha_q h_q \quad (7.19)$$

as our iterative process. h_q is essentially a choice of direction along which we go from the q^{th} estimate to the $q+1^{\text{th}}$ estimate; α_q is how far we go in that direction. To use (7.19), we must have a program for selecting α_q and h_q .

Newton's method for finding the minimum of a function of n variables is basically written in the form (7.19). If

$$F(\alpha) = F(x + \alpha h) = F(x) + \alpha F'(x, h) + \frac{\alpha^2}{2} F''(x, h), \quad (7.20)$$

truncating the Taylor Series at second order terms, then we minimize the right hand side with respect to α and obtain

$$\alpha = -\frac{F'(x, h)}{F''(x, h)},$$

and hence we take

$$\alpha_q = -\frac{F'(x_q, h_q)}{F''(x_q, h_q)}. \quad (7.21)$$

The Method of Conjugate Gradients (Hestenes, M.R., and E. Steifel, Jnl. of Research, Nat. Bur. Stds., RP2379, V.49, 1952) is a variant of (7.19). For a discussion of iterative methods for linear systems, see Hayes, Nat. Bur.

Stds., Report No. 1733, 1955.

The gradient method, as applied to the problem of minimizing an integral, is discussed in RAND Report, RM102, 1949, by M.R. Hestenes.

Let us discuss in some detail an iterative method for finding the minimum of a function of n variables $F(x) = F(x^1, x^2, \dots, x^n)$. By iterating on x , we hope to improve this approximation of the solution by choosing a δx such that

$$\tilde{x} = x + \delta x = x + \alpha h,$$

i.e., $x_{q+1} = x_q + \alpha_q h_q$. Program for α :

- a) $\alpha_q = \alpha = \text{constant}$. Usually if α is chosen too large for convergence, choose $\alpha^* = \alpha/2$; if too small, choose $\alpha^* = 2\alpha$. One can also step α to find the value for quickest convergence.

b)

$$\alpha_q = -\beta \frac{F'(x_q, h_q)}{F''(x_q, h_q)},$$

where we have added a scale factor β ($1 - \epsilon \leq \beta \leq 1 + \epsilon$). If $\beta < 1$, one says he is under-relaxing; if $\beta > 1$, over-relaxing.

Program for h_q :

- a) Choose n linearly independent vectors

$$u_1, u_2, \dots, u_n.$$

Let h_q be some combinations, say

$$h_q = (u_1, u_2, \dots, u_n, u_1, \dots).$$

This is the usual Gauss-Seidel procedure. Any combination of the u_j can be made.

$$b) \quad h_q = -\text{grad } F. \quad \text{Recall } F'(x_0, h) = F_{x^1} h^1 = \text{grad } F \cdot h.$$

Usually we define the dot product of the two vectors x, y as

$$x \cdot y = x_1 y_1.$$

We could define

$$x \cdot y = \sum g_{1j} x_1 y_j,$$

a positive definite form. Then, for quick convergence, we could write

$$(\text{grad } F)_1 = g^{1j} \frac{\partial F}{\partial x^j},$$

where $g_1, g^{jk} = \delta_1^k$, and then choose g^{1j} so that $\text{grad } F$ points toward the minimum point, not normal to $F = \text{constant}$ as is usually the case. This implies a particular choice of h_q .

Newton's method appears in all phases of numerical analysis. When solving the equation

$$G(x) = 0_j,$$

we write

$$G(x + \delta x) = G(x) + G'(x) \delta x,$$

set the right hand side equal to zero, and pick

$$\delta x = -\frac{G(x)}{G'(x)}.$$

For a system of equations $G_1(x) = 0$, we put

$$0 = G_1(x + \delta x) = G_1(x) + \frac{\partial G_1}{\partial x^j} \delta x^j.$$

Then we put $g_{1j}(x) = \frac{\partial G_1}{\partial x^j}$, not necessarily a symmetric matrix, and then let

$$\delta_{x^j} = -g^{jk} G_k(x).$$

If $G_1 = \frac{\partial F}{\partial x^1}$, then $g_{1j} = \frac{\partial F}{\partial x^1 \partial x^j}$ is symmetric.

For quadratic functions F , we choose $(\text{grad } F)_1 = g^{1j} \frac{\partial F}{\partial x^j}$;

but if F is of an indefinite form, we may get saddle points.

Suppose we are solving a minimization problem with constraints

$$f(x) = \min$$

$$g(x) = 0.$$

The first necessary condition is

$$f_{x^1} + \lambda g_{x^1} = 0.$$

By Newton's method,

$$f_{x^1} + \lambda g_{x^1} + (f_{x^1 x^j} + \lambda g_{x^1 x^j}) \delta x^j + \delta \lambda g_{x^1} = 0,$$

and

$$g + g_{x^j} \delta x^j = 0.$$

To solve these equations for δx^j and $\delta \lambda$, we must have

$$\begin{vmatrix} f_{x^1 x^j} + \lambda g_{x^1 x^j} & g_{x^1} \\ g_{x^j} & 0 \end{vmatrix} \neq 0,$$

and then to iterate, we put

$$\begin{aligned}x_{q+1}^j &= x_q^j + \delta x^j \\ \lambda_{q+1} &= \lambda_q + \delta \lambda_q.\end{aligned}$$

Finally, let us consider Newton's method for finding the solution of a simple differential equation

$$T = 1 + y'^2 - yy'' = 0 \quad \text{subject to } y(a) = A \\ y(b) = B$$

with $y > 0$

hence $y'' > 0$.

This is the catenary problem. To solve this, guess a function $y(x)$ to satisfy the boundary conditions, and set $T + \delta T = 0$, i.e.,

$$T + 2y'\delta y' - \delta yy'' - y\delta y'' = 0$$

with $\delta y(a) = 0$, $\delta y(b) = 0$.

Improve on the guess by solving this linear equation for δy .

This method is also applicable to simple and multiple integrals.

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Rendezvous Problems

by

J.C. Houbolt

Rendezvous Problems

These talks will be roughly divided into the following topics:

1. Definitions
2. Ascent trajectories
3. Parking orbits
4. Mission analysis
5. Rendezvous in planetary travel

Definitions

The definitions of the major symbols are:

- a: semi-major axis
- E: eccentric anomaly
- V: true anomaly
- p: parameter of ellipse - semilatus rectum
- T: period
- i: inclination angle
- ω : argument of perigee
- Ω : argument of ascending node
- e: eccentricity

In terms of the quantities we will use the well-known relations:

1. $r_p = a(1 - e)$; pericenter distance
2. $r_a = a(1 + e)$; apocenter distance
3. $p = a(1 - e^2)$; semilatus rectum
4. $T = 2\pi a^{3/2} \mu^{-1/2}$,

$\mu = 1.407639 \times 10^6 \text{ ft}^3/\text{sec}$ for the Earth's gravitational constant

$$5. \quad n = \frac{2\pi}{T} = \mu^{1/2} a^{-3/2} ; \text{ mean motion}$$

$$6. \quad M = n(t - T) ; \text{ mean anomaly}$$

$T = \text{time at epoch.}$

If the Earth's potential function is represented by

$$7. \quad U = \frac{\mu}{r} \left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r} \right)^n P_n(\sin L) \right]$$

where

$P_n =$ Legendre polynomial of order n

$L =$ Latitude angle,

then the first order secular perturbations in the orbital elements of an Earth satellite in the absence of air drag are:

$$8. \quad \dot{\Omega}_s = - \frac{3}{2a} \sqrt{\frac{\mu}{a}} J_2 \left(\frac{R}{p} \right)^2 \cos i \text{ rad/sec} ; R = \text{equatorial radius}$$

$$9. \quad \dot{\omega}_s = \frac{3}{4a} \sqrt{\frac{\mu}{a}} J_2 \left(\frac{R}{p} \right)^2 (-1 + 5 \cos^2 i) \text{ rad/sec}$$

$$10. \quad \dot{M}_s = \frac{3}{4a} \sqrt{\frac{\mu}{a}} J_2 \left(\frac{R}{p} \right)^2 \sqrt{1 - e^2} (-1 + 3 \cos^2 i) \text{ rad/sec}$$

$$11. \quad \dot{\Omega}_s = - 3 \pi J_2 \left(\frac{R}{p} \right)^2 \cos i \text{ rad/rev.}$$

$$12. \quad \dot{\omega}_s = 3 \pi J_2 \left(\frac{R}{p} \right)^2 (-1/2 + 5/2 \cos^2 i) \text{ rad/rev}$$

$$13. \quad \dot{M}_s = 3 \pi J_2 \left(\frac{R}{p} \right)^2 \sqrt{1 - e^2} (-1/4 + 3/4 \cos^2 i) \text{ rad/rev}$$

where $J_2 = 1082.28 \times 10^{16}$.

Example:

For an orbit with $i = 30^\circ$ and an altitude of 300 statute miles, one finds that

$$\dot{\Omega} = -0.442^\circ/\text{rev.} \approx -6.8^\circ/\text{day}$$

$$\dot{\omega} = 0.705^\circ/\text{rev.} \approx 10.8^\circ/\text{day}$$

Rendezvous Phases

Rendezvous can be divided into the three phases

- i. ascent of injection into transfer orbit
- ii. terminal phase
- iii. docking - contact between ferry and target vehicles.

There are a wide variety of possible types of ascent maneuvers and a few remarks will be made concerning the characteristics of some of the basic types of ascent maneuvers.

a. In - plane ascent:

An in - plane ascent requires that the target vehicle travel in a compatible orbit; that is, an orbit in which the target passes over the launch site at least once per day. This is a severe requirement and its practical realization will probably require means for adjusting the orbital period of the target vehicle.

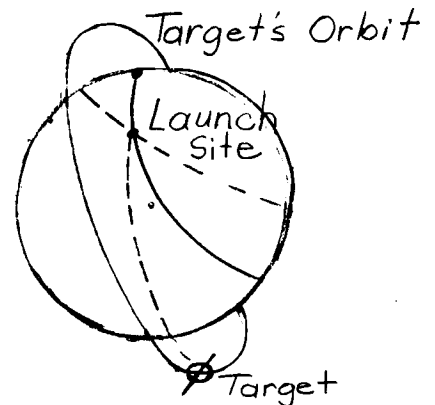


Figure 1
Transfer Orbit

b. Adjacency transfer:

The ferry is inserted into an orbit close to that of the target, but not necessarily in the same orbital plane. The ferry transfer orbit is selected so that its orbit is coaltitude and has the same velocity as the target at the time at which the two orbits intersect. At the time of orbit intersection, the ferry is given a velocity impulse such that its orbit plane is made coincident with that of the target.

c. Two - impulse transfer:

The first impulse inserts the ferry into a transfer orbit such that the apogee of the transfer occurs at the orbit of the ferry and the timing is such that the ferry and target are simultaneously at the apogee of the transfer orbit. When the two orbits touch, a second velocity impulse is given to the ferry to bring it up to orbital speed and, if necessary, change its orbital plane to coincide with that of the target.

d. General ascent:

The ferry is injected into a general transfer orbit which is required to intersect the target on either the outgoing leg or the incoming leg. The timing problem for these ascents is very critical and typical launch windows are only of about 3 minutes in duration.

e. Parking orbits:

An intermediate parking orbit greatly simplifies the timing problems for an ascent transfer trajectory. The ferry is first launched into a circular orbit at a lower altitude than that of the target. Because the ferry will have a shorter period of revolution, it will gain

on the target with respect to their geocentric angles. At the proper time, the ferry is given a velocity impulse into a transfer orbit which will bring it into position for the final rendezvous maneuver.

Velocity Penalty for Maneuvers

a. Equal velocities, in - plane maneuvers:

Suppose that the interceptor (ferry) and the target vehicles have the same velocity magnitude but different directions; Figure 2.

For small α ,

$$14. \quad \Delta v = \alpha v.$$

For a typical velocity of 25,000 ft/sec, the velocity increment required per degree separation of the paths would be of the order

$$15. \quad \Delta v = \frac{\pi}{180} \times 25 \times 10^3 = 436 \text{ ft/sec.}$$

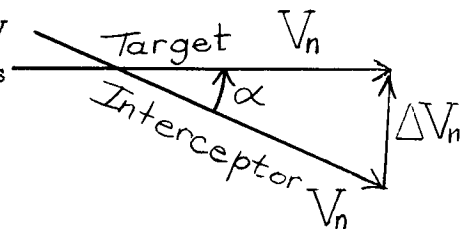


Figure 2

This is a costly maneuver as measured in units of required velocity impulse.

b. Two - impulse maneuver:

From Figure 2,

$$16. \quad v_2^2 = v_1^2 + v_0^2 - 2v_1v_0 \cos \alpha.$$

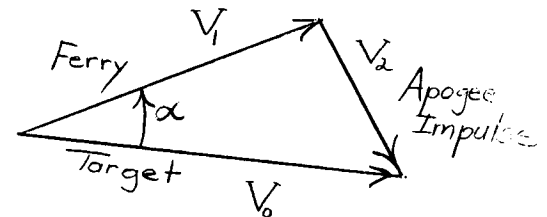


Figure 3

The velocity penalty for the plane change is

$$17. \quad \Delta v = v_1 + v_2 - v_0,$$

for small α , such that $\sin \alpha \approx \alpha$, 16. and 17. yield

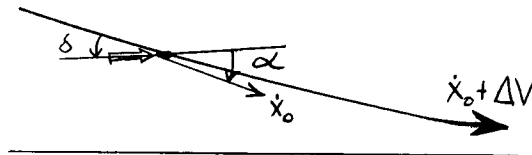
$$18. \quad \Delta v = \frac{v_1 v_0}{2(v_0 - v_1)} \alpha^2.$$

Example:

Typical numbers at apogee are $v_1 = 10 \times 10^3$ ft/sec, $v_2 = 15 \times 10^3$, $v_0 = 25 \times 10^3$ ft/sec. If $\alpha = 5.7^\circ$, then $\Delta v \approx 83$ ft/sec. Thus the two-impulse maneuver is less costly than the previous case. The economy partly comes from the fact that the velocity impulse can correct the interceptor's speed at the same time that its orbital plane is shifted.

c. Dog - leg maneuvers:

Dog - leg maneuvers during thrusting may also be used during ^{ascent or} transfer trajectories to effect an orbital plane shift. Thrust is made in the transverse direction by tilting the rocket thrust by an angle δ from the vehicle's flight path. Let



Δv = required increment of velocity.

Figure 4

It can be demonstrated that if $\dot{y} < \dot{x}_0$, then for δ held constant,

$$19. \quad \frac{\dot{y}}{\dot{x}_0} = \alpha = \frac{\Delta v}{\dot{x}_0} \delta$$

Example:

If $\dot{x}_0 = 10 \times 10^3$ ft/sec, $\Delta v = 15 \times 10^3$ ft/sec, one finds that $\alpha = 1.5 \delta$.

Thus the dog - leg maneuver can change the angle of the trajectory plane on the same order as the rocket motor gimbel angle used, and with minor penalty on the forward acceleration.

General Direct Ascent

a. General Direct Ascent

The rendezvous window is defined as the interval of time on the launch pad during which a rendezvous ascent can be made without an "excess" fuel penalty.

It has been established that Hohmann - type transfers produce minimum energy transfer. Soft - rendezvous is the situation in which the speed and orbit direction are the same for both the interceptor and target vehicles. A Hohmann - type transfer can be used if the target is at A_L at interceptor launch (ahead of insertion point). The intercept takes place at A_R .

The general cases occur for the target at either B_L (leading) or C_L (lagging) with the intercept accomplished at the intersection points B_R or C_R , respectively.

One can investigate the maximum spread in angle between initial points B_L and C_L which determine the allowable launch window with a restriction on the available ΔV capability of the interceptor.

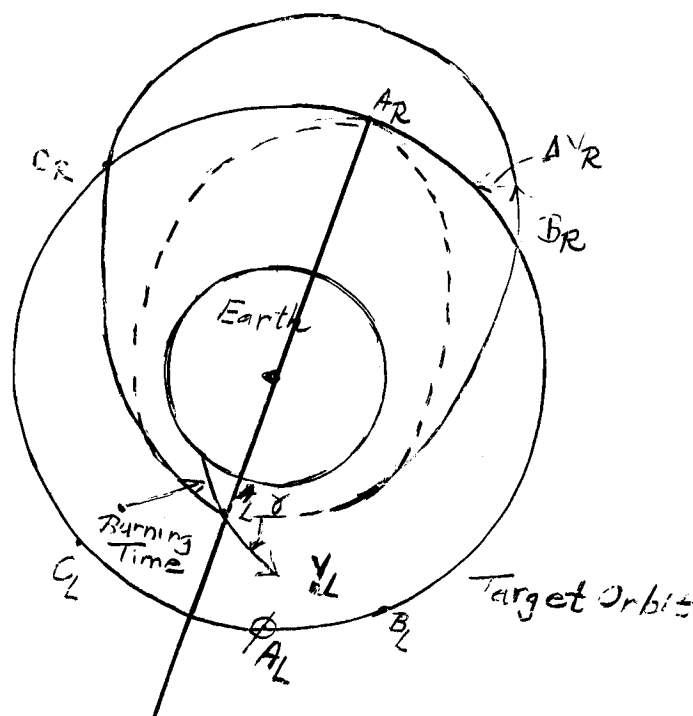


Figure 5
Transfer Orbits

Suppose that the total vehicle thrust capability is

$$20. \quad v_L + \Delta v_R = 27,000 \text{ ft/sec.}$$

One can show that the launch window shown in Figure 6, is -7.4 to 6.1° or roughly 13° , which corresponds to about 3 minutes for typical orbits. If the thrust capability is increased to 3×10^4 ft/sec, the launch window increases to about 15 minutes.

It is thus seen that the launch window is very sensitive to the total vehicle capability.

b. Indirect ascent schemes

Parking orbits can be employed to extend the launch windows from the order of minutes to hours.

Suppose, as shown in Figure 7, that the inclination of the target's orbital plane is only slightly larger than the latitude i_L of the launch site. Further, suppose that the interceptor is launched in a close orbit. That is, only a small angle change is required for rendezvous. Let Δi denote the required difference in the inclination of the orbital plane. It can be demonstrated that

$$21. \quad \cos \theta = \frac{\sin i_L \cos i_0 - \sin \Delta i}{\cos i_L \sin i_0}.$$

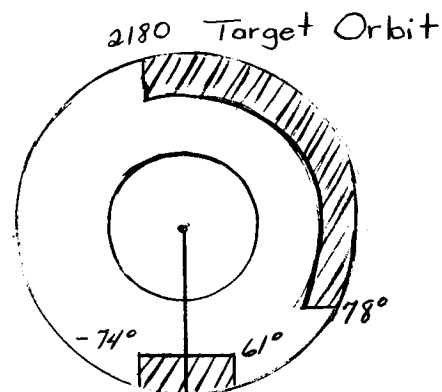


Figure 6
Launch Windows

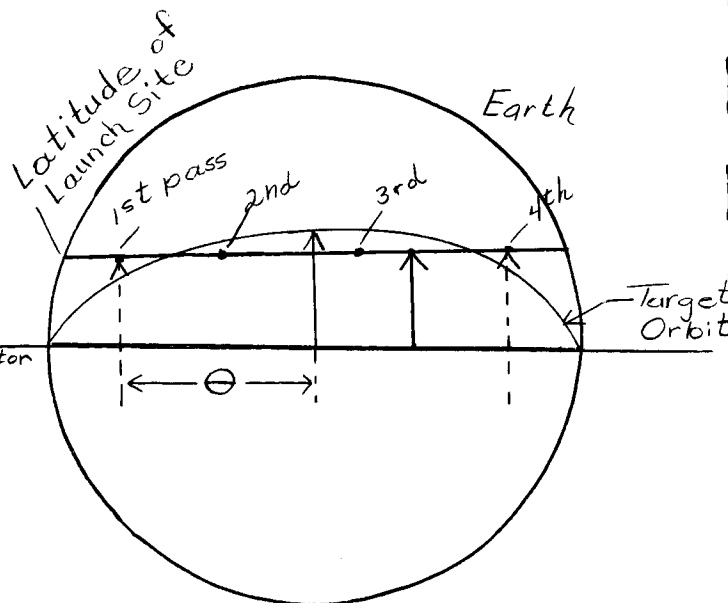


Figure 7

Example:

If $i_L = 28^\circ$ (Cape Canaveral)

i_0	Δi	θ
30°	2°	32.6°
30.4	2.4	36.0
31.0	3.0	39.5

Next consider two types of transfer orbits:

Case a: Transfer apogee at target height (Gemini Program Maneuver)

A chasing orbit is obtained by launching a transfer such that the apogee is tangent to the target's orbit. Thus the ferry or interceptor gains on the target during each revolution until a constellation is attained for which a single small impulse is sufficient to effect the rendezvous.

Let: θ = angular difference
 ν = number of revolutions required to overcome θ deficiency

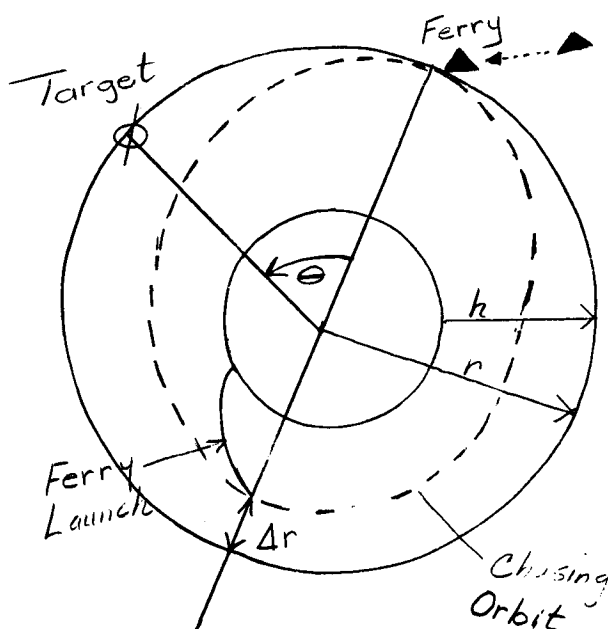


Figure 8

Hohmann - type Transfer

One can verify that

$$21. \quad \frac{\Delta r}{r} = \frac{4}{3} \frac{\theta}{360 \nu} ,$$

$$22. \quad \frac{\Delta v}{v_0} = \frac{1}{3} \frac{\theta}{360 \nu} ,$$

where V_0 is the orbital speed.

Example:

If $\theta = 20^\circ$

$\nu = 1$

$r = 4260$,

then $\Delta r = 315$. This cannot be accomplished in one revolution because Δr is greater than target altitude, here considered to be 300 s.m. Therefore, let $\nu = 2$, and then $\Delta r = 158$ miles and $\Delta v = 213$ ft/sec.

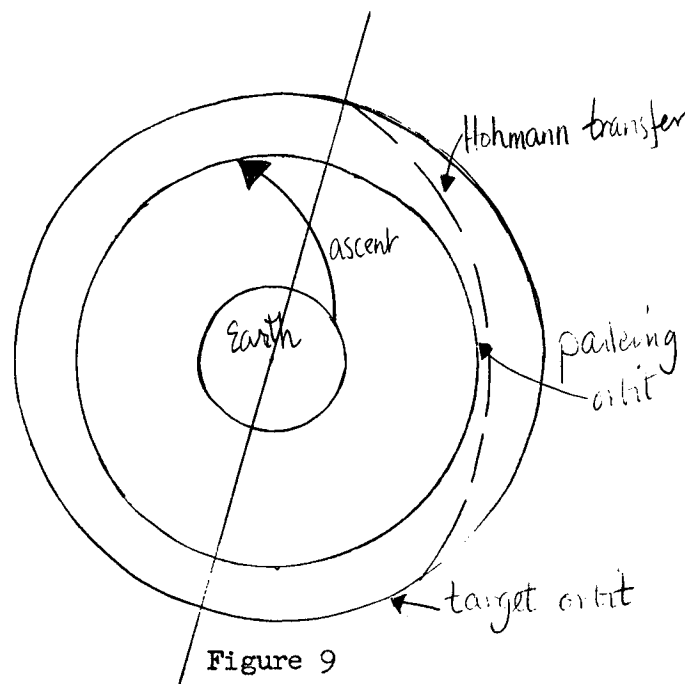
Case b: Parking orbit

For an intermediary parking orbit, 21. is modified to

$$23. \quad \frac{\Delta r}{r} = \frac{2}{3} \frac{\theta}{360(\nu - 3/4)}.$$

Thus the basic technique in the use of chasing or parking orbits is to launch the ferry any time it is ready during the time interval the launch site is close to the orbital plane of the target,

Figure 7. From this figure and the table relating Δi and θ , this may be in the interval of several successive orbital passes. Any geocentric angular deficiency that the ferry may have is made up by use of the chasing or parking orbit. It is seen that the holding back for subsequent addition of a rendezvous velocity increment ΔV_R allows this type rendezvous to be made at substantially the same characteristic velocity increment as would be involved in a direct ascent rendezvous. These indirect schemes provide for launch windows up to 3-5 hours, instead of minutes.



Terminal Phase

Terminal phase starts when ferry is about 50 miles from the target.

Two types of terminal maneuvers are usually considered.

(i) Proportional navigation: maintain line-of-sight fixed in inertial space; or, maintain zero angular rate.

(ii) Orbital mechanics: compute coast orbits of target and ferry to determine if they intersect. If no intersection, compute required change in ferry orbit to produce orbit intersection.

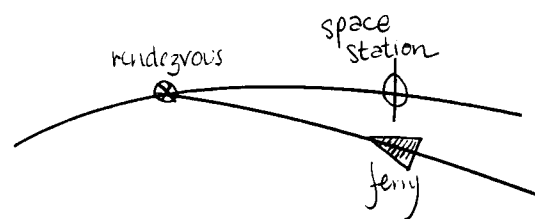


Figure 10

(a) Terminal Guidance

The terminal guidance equations for a variety of assumed models are given in Table 1. As an example, consider the equations in rotating rectangular coordinates for a model having a spherical earth, circular target orbit, "zero-order" gravity. The equations of motion are

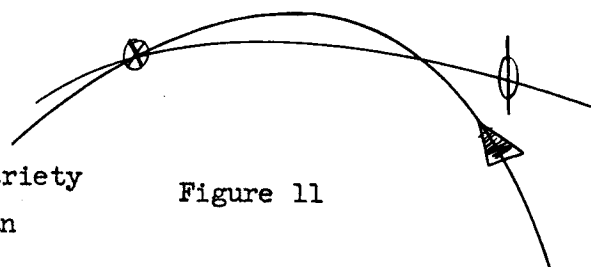


Figure 11

$$24. \quad \ddot{x} - 2\omega \dot{y} = \frac{T_x}{m}$$

$$\ddot{y} + 2\omega \dot{x} - 3\omega^2 y = \frac{T_y}{m}$$

$$\ddot{z} + \omega^2 z = \frac{T_z}{m}$$

Assume no thrust, $T_x = T_y = T_z = 0$, then the solutions to 24. are

$$25. \quad x = \left(x_0 + \frac{2\dot{y}_0}{\omega}\right) + (-3\dot{x}_0 + 6\omega y_0)t - 2\left(3y_0 - 2\frac{\dot{x}_0}{\omega}\right) \sin \omega t - \frac{\dot{y}_0}{\omega} \cos \omega t,$$

$$26. \quad y = (4y_0 - 2\frac{\dot{x}_0}{\omega}) + (-3y_0 + 2\frac{\dot{x}_0}{\omega}) \cos \omega t + \frac{\dot{y}_0}{\omega} \sin \omega t,$$

$$27. \quad z = a_1 \sin \omega t + b_1 \cos \omega t,$$

where $x_0, \dot{x}_0, y_0, \dot{y}_0$ are the initial conditions.

The general relative motion seen be the ferry in these coordinates is shown in Figure 12. The ellipse is centered at the target and has the following parameters:

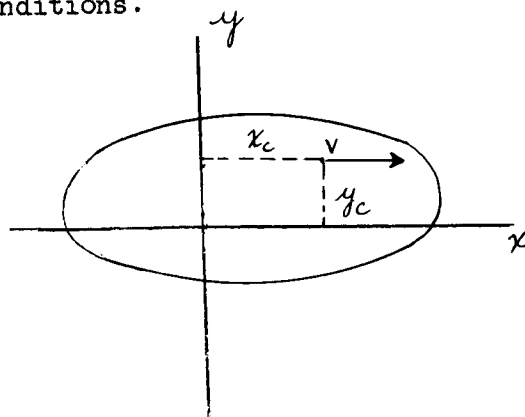


Figure 12

$$28. \quad v = -3\dot{x}_0 + 6\omega y_0$$

$$x_c = x_0 + 2\frac{\dot{y}_0}{\omega}$$

$$y_c = 4y_0 - 2\frac{\dot{x}_0}{\omega}$$

$$a = 2b$$

$$b = \left[\left(\frac{\dot{y}_0}{\omega} \right)^2 + \left(3y_0 - 2\frac{\dot{x}_0}{\omega} \right)^2 \right]^{\frac{1}{2}}.$$

Suppose that $v = y_c = 0$; this implies that

$$29. \quad \dot{x}_0 = 2\omega y_0,$$

which is the condition for which the orbital period of the transfer orbit is equal to that of the target.

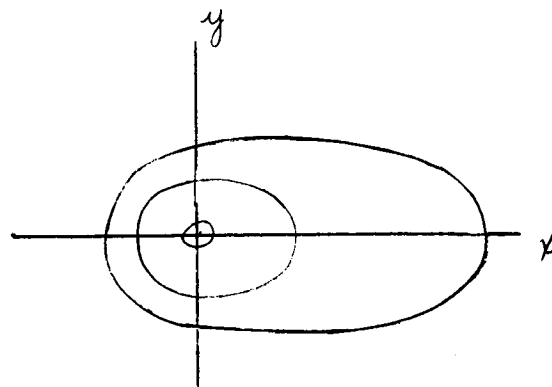


Figure 13

If

$$30. \quad \dot{x}_0 = 2\omega y_0$$

$$\dot{y}_0 = -\frac{\omega x_0}{2},$$

we have the situation illustrated in Figure 13, in which the ellipse is centered about the target.

If the target is itself in elliptical motion, it can be shown that the same form of terminal equations apply to the relative motion.

(b) Two Impulse Terminal Phase - Orbital Mechanics Scheme

Let $\omega/2\pi$ be the period of the target and t_r denote the time interval required to effect a rendezvous. Figures 15 and 16 illustrate the effect of the parameters required for a rendezvous.

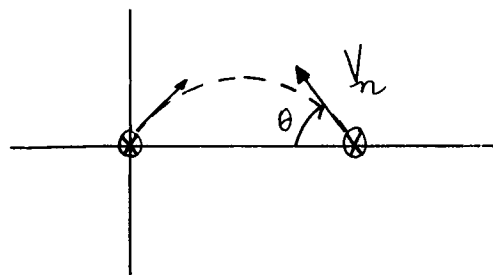


Figure 14

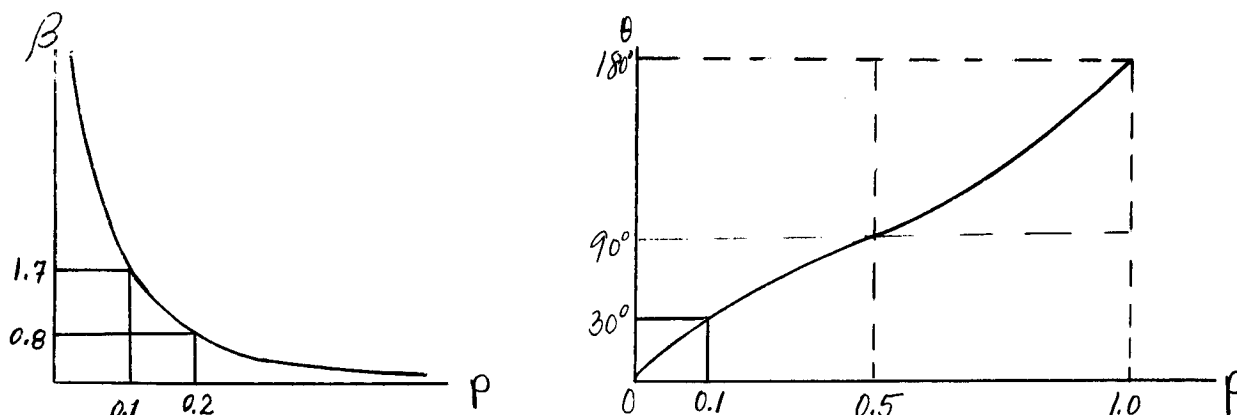


Figure 15

$$31. \quad p = \frac{\omega t_r}{2\pi}, \text{ period ratio of the orbits,}$$

ω = angular rate of orbit

$$v = \beta \omega x_0.$$

Example:

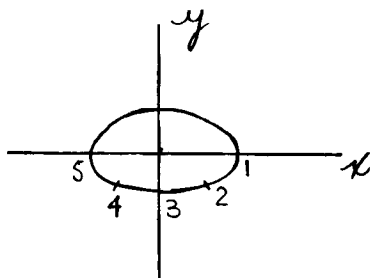
If the altitude $h \sim 200$ miles, $\omega \approx 0.00114$.

For $\dot{x}_0 = 5000$ ft.

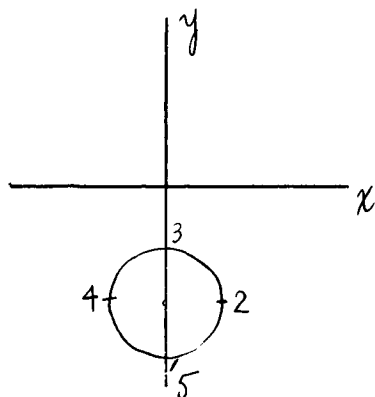
$$v = 5.7 \beta$$

If $\beta \approx 2$, $v \approx 10$ ft/sec to complete a rendezvous in a time $t_r = 10$ minutes.

(i) rotating axis system



(ii) inertial fixed system



Ferry Behind Space Station

(c) Proportional Navigation

Let R be the line-of-sight distance between the station and the vehicle. For proportional navigation, an intercept occurs if

$$32. \quad \dot{R}^2 = 2aR$$

a = acceleration.

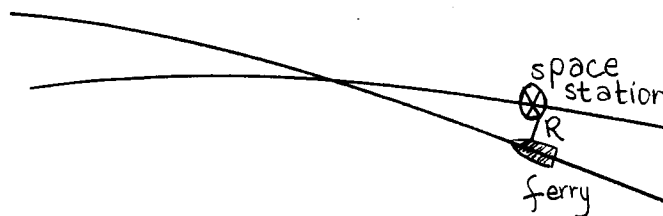
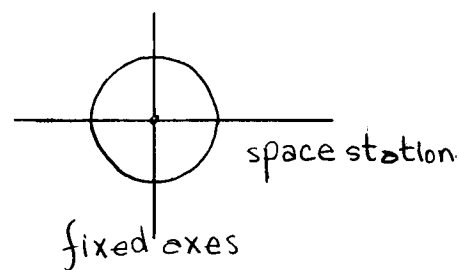
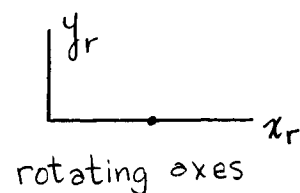
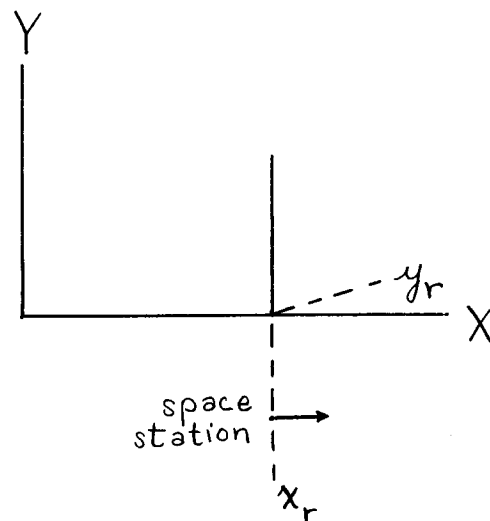


Figure 16

One maneuver for "braking" into a rendezvous is shown in Figure 17. A thrust is applied at the "on" line and removed at the "off" line. The vehicle then coasts until the "on" line is again met. The rendezvous is then made by "braking" in this stepwise fashion.

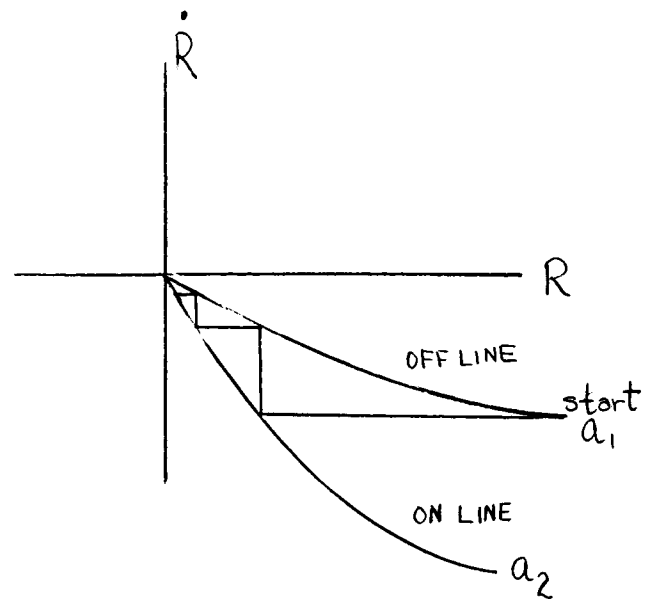


Figure 17

Mission Analysis

Mission analysis is used for booster design, or specifying the rocket thrust capabilities. As an example of mission design, consider a comparison of two types of lunar mission profiles:

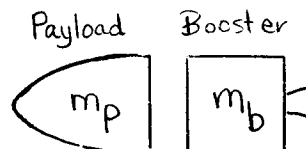
- (i) direct ascent
- (ii) direct ascent with rendezvous in parking orbit about moon

The basic rocket equation can be written as

$$32a. \quad \frac{m_0}{m} = e^{\frac{\Delta v}{u}} = K$$

$$u = Ig$$

I = specific impulse



Consider the vehicle configuration of Figure 18.

Apply 32a. to obtain

Figure 18

$$33. \frac{m_p + m_b}{m_p + \epsilon m_b} = K,$$

where ϵm_b = burn out weight of booster.

Solve for

$$34. m_b = \frac{K - 1}{1 - \epsilon K} m_p = r m_p.$$

The required total weight is then

$$35. m_T = m_p + m_b = \frac{(1 - \epsilon)K}{1 - \epsilon K} m_p = R m_p.$$

(a) Direct Ascent to Moon and Return

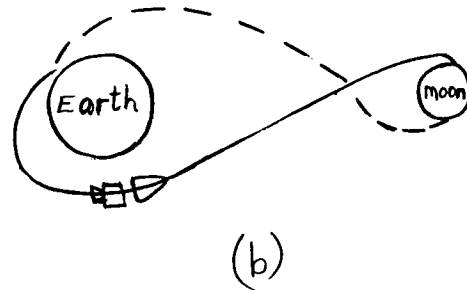
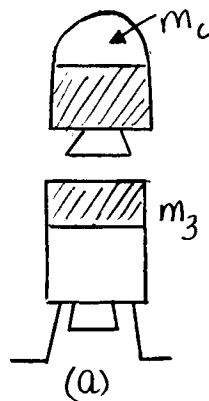


Figure 19

The sequence is

(i) Land on moon by means of m_3 ; ΔV_3 is the required velocity increment.

(ii) From the moon's surface, launch m_c to Earth return, ΔV_4 is the velocity increment.

Using the preceding equations, the mass that must be used to escape the Earth is

$$36. \quad M_e = R_3 (m_T + m_3) \quad ; \quad m_T = R_4 m_c = R_3 (R_4 m_c + m_3)$$

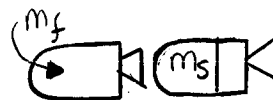
Typical velocity increments are:

$$\Delta v_3 = 10,640 \text{ ft/sec}$$

$$\Delta v_4 = 10,330 \text{ ft/sec.}$$

(b) Lunar Rendezvous

The mission sequence is



(i) Decelerate the vehicle into a moon orbit; Δv_1

Figure 20

(ii) Descend to the moon with m_f , ascend to rendezvous; $\Delta v_d, \Delta v_a$.

(iii) Return to Earth; Δv_2

One finds

$$37. \quad m = R_a m_f,$$

$$38. \quad m_L = R_d (m + m_s) = R_d (R_a m_f + m_s).$$

It can be verified that the mass, m_e , that escapes from the Earth is

$$39. \quad m_e = R_{12} \left(\frac{m_L}{R_2} + m_c \right).$$

Typical velocity increments are:

$$\Delta v_a = 6800 \text{ ft/sec}$$

$$\Delta v_d = 6800$$

$$\Delta v_2 = 3530$$

$$\Delta v_{12} = \Delta v_1 + \Delta v_2 = 7370.$$

It is interesting to compare the two types of lunar profile missions. For a direct ascent with typical values

$$40. \quad M_e = 10 m_c + 2.745 m_s$$

and

$$41. \quad m_e = 2.435 m_c + 8.96 m_f + 3.81 m_s$$

$$42. \quad m_L = 5.52 m_f + 2.35 m_s .$$

If $m_c = 13,000 \text{ lb}$, $m_f = 3500$, $m_s = 0$; one finds that

$$M_e = 130,000 \text{ lbs}$$

$$m_e = 64,000$$

$$m_L = 19,000.$$

These figures indicate the economy of a lunar rendezvous mission as compared to a direct ascent.

Rendezvous in Interplanetary Transfer

Rendezvous problems for interplanetary flights are exactly similar to those already discussed except in the near vicinity of the departing and destination planets. Figure 21 illustrates the hyperbolic escape

orbit in the vicinity of the Earth.

The escape velocity is computed from

$$43. \quad v^2 = \mu \left(\frac{2}{r} + \frac{1}{a} \right)$$

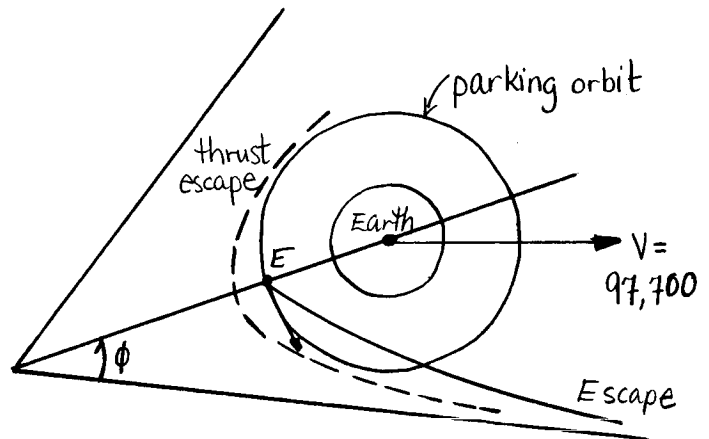
or, in equivalent form

$$44. \quad v_E^2 = 2v_0^2 + v_\infty^2, \quad ,$$

where

v_0 = circular velocity at height r

v_∞ = hyperbolic excess velocity.



$$45. \quad \tan \phi = \frac{v_E v_\infty}{v_0^2}.$$

Figure 21

The thrust required for escape is computed from the Lagrangian

$$46. \quad L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{\mu}{r},$$

and

$$47. \quad \delta \omega = F_r \delta r + F_\theta r \delta \theta.$$

The equations of motion for a thrusting escape are

$$48. \quad \ddot{r} = r\dot{\theta}^2 + \frac{\mu}{mr^2} = \frac{\mu m}{m} \frac{\dot{r}}{r}$$

$$49. \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{\mu m}{m} \frac{r\dot{\theta}}{V}$$

$$V = (\dot{r}^2 + r^2 \dot{\theta}^2)^{\frac{1}{2}}.$$

Rendezvous Problems

Instead of a Hohmann transfer, a faster orbit can be used as shown in Figure 22. The following table indicates the characteristics of a minimum energy Hohmann transfer to Mars compared to a possible fast orbit.

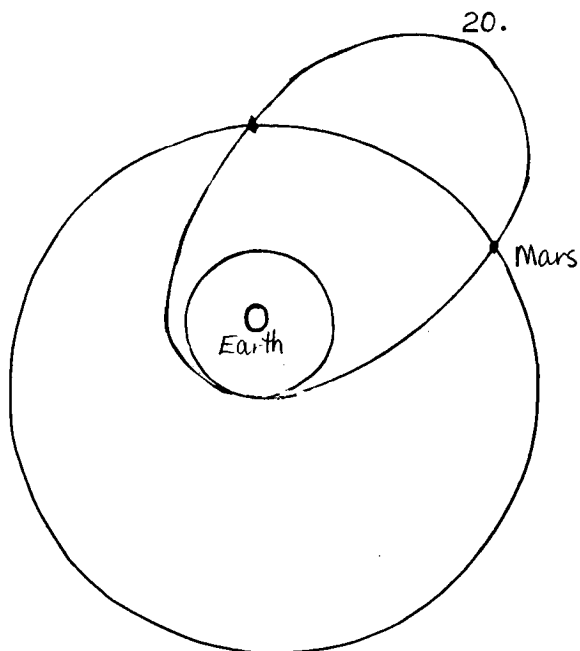


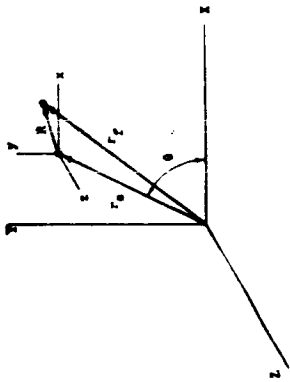
Figure 22

	<u>Mars Stay Time</u>	<u>Travel Time</u>	<u>Total Time</u>	<u>$\sum \Delta v$</u>
Hohmann	460 days	520 days	980 days	36,600 ft/sec
Fast Orbit	30	290	320	76,000

Reference:

J.C. Houbolt, "Problems and Potentialities of Space Rendezvous,"
Astronautica Acta, Volume VII, Fasc. 5-6, 1961.

Table 1. Terminal Guidance Equations

Inertially fixed axes		
		
	Vector form	Rectangular coordinates
Exact	$\frac{\partial^2 \vec{r}_f}{\partial t^2} - \nabla \frac{\mu}{r_f} = \frac{\vec{T}}{m}$	<p>Similar to form immediately below except in the expansion of $\nabla (\mu/r_f)$</p>
Spherical earth	$\frac{d^2 \vec{r}_f}{dt^2} + \frac{GM}{r_f^3} \vec{r}_f = \frac{\vec{T}}{m}$ $\vec{r}_f = \vec{r}_s + \vec{h}$ $\vec{r}_s = (X, Y, Z)$ $= (r_s \cos \theta, r_s \sin \theta, 0)$ $\vec{R} = (x, y, z)$	$\ddot{x} + (\ddot{r}_s - r_s \dot{\theta}^2) \cos \theta - (2 \dot{r}_s \dot{\theta} + r_s \ddot{\theta}) \sin \theta + \frac{GM}{r_f^3} (x + r_s \cos \theta) = \frac{T_x}{m}$ $\ddot{y} + (\ddot{r}_s - r_s \dot{\theta}^2) \sin \theta + (2 \dot{r}_s \dot{\theta} + r_s \ddot{\theta}) \cos \theta + \frac{GM}{r_f^3} (y + r_s \sin \theta) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_f^3} z = \frac{T_z}{m}$

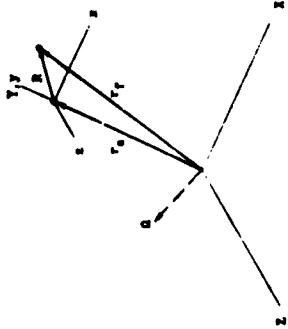
Rotating set of axes		
		
Comments	Vector form	Rectangular coordinates
$\vec{r}_I = \vec{r}_s + \vec{R}$ μ/r_I is the gravity potential due to earth, moon, planets etc.	$\frac{\partial^2 \vec{r}_I}{\partial t^2} + 2\vec{\Omega} \times \frac{\partial \vec{r}_I}{\partial t} + \vec{\Omega} \times \vec{r}_I + \vec{\Omega} \times \vec{\Omega} \times \vec{r}_I - \vec{r}_I \frac{\mu}{r_I} = \frac{\vec{T}}{m}$	Similar to form immediately below except in the expansion of $V(\mu/r_I)$
θ is the angular velocity of station about center of earth, r_s is radial position of station	$\frac{d^2 \vec{r}_I}{dt^2} + 2\vec{\Omega} \times \frac{d\vec{r}_I}{dt} + \vec{\Omega} \times \vec{r}_I + \vec{\Omega} \times \vec{\Omega} \times \vec{r}_I + \frac{GM}{r_s^3} \vec{r}_I = \frac{\vec{T}}{m}$ $\vec{r}_I = (x, y + r_s, z) = \vec{r}_s + \vec{R}$ $\vec{r}_s = (0, r_s, 0)$ $\vec{R} = (x, y, z)$ $\vec{\Omega} = (0, 0, \dot{\theta})$	$\ddot{x} - (y + r_s)\ddot{\theta} - 2(\dot{y} + \dot{r}_s)\dot{\theta} - x\left(\dot{\theta}^2 - \frac{GM}{r_s^3}\right) = \frac{T_x}{m}$ $\ddot{y} + x\ddot{\theta} + 2\dot{x}\dot{\theta} + \ddot{r}_s - (y + r_s)\left(\dot{\theta}^2 - \frac{GM}{r_s^3}\right) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_s^3}z = \frac{T_z}{m}$ <p>EGGLESTON [4, 5]</p>

Table I (continued)

	Vector form	Rectangular coordinates
Spherical earth, Station in a circular orbit	$\frac{d^2 \vec{R}}{dt^2} = -\frac{GM}{r^3} \vec{r}$	$\ddot{x} = -\frac{GM}{r^3} (x + r_s \cos \theta) = -\frac{T_x}{m}$ $\ddot{y} = -\frac{GM}{r^3} (y + r_s \sin \theta) = -\frac{T_y}{m}$ $\ddot{z} = -\frac{GM}{r^3} z = -\frac{T_z}{m}$
Spherical earth, Circular orbit, 1st order gravity field	$\frac{d^2 \vec{R}}{dt^2} = -\frac{GM}{r_s^3} \left(\vec{R} - 3 \frac{\vec{r}_s \cdot \vec{R}}{r_s^2} \vec{r}_s \right) = -\frac{\vec{T}}{m}$ HORD [9], KURHJEN [10], BRISSENBEN [11]	$\ddot{x} + \omega^2 (x - 3y \cos^2 \theta - 3y \sin \theta \cos \theta) = -\frac{T_x}{m}$ $\ddot{y} + \omega^2 (y - 3x \sin \theta \cos \theta - 3y \sin^2 \theta) = -\frac{T_y}{m}$ $\ddot{z} + \omega^2 z = -\frac{T_z}{m}$
Spherical earth, Circular orbit, "Zero-order" gravity		
No gravity	$\frac{d^2 \vec{R}}{dt^2} = \frac{\vec{T}}{m}$ any orbit HORD [9]	Circular orbit $\ddot{x} = \frac{T_x}{m}$ $\ddot{y} = \frac{T_y}{m}$ $\ddot{z} = \frac{T_z}{m}$

Comments	Vector form	Rectangular Coordinates
$r_s = \text{Constant}$ $\dot{\theta} = \text{Constant} = \omega$ $\dot{\Omega} = \text{Constant} = (0, 0, \omega)$ $\omega^2 = \frac{GM}{r_s^3} = \dot{\theta}^2$	$\frac{d^2 \vec{R}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{R}}{dt} + \vec{\omega} \times \vec{\omega} \times \vec{R} + \frac{GM}{r_s^3} \vec{r}_f = \frac{\vec{T}}{m}$ <p>or</p> $\frac{d^2 \vec{R}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{R}}{dt} + \omega^2 \vec{R} + \left(\frac{GM}{r_s^3} - \omega^2 \right) \vec{r}_f = \frac{\vec{T}}{m}$	$\ddot{x} - 2\omega \dot{y} - x \left(\omega^2 - \frac{GM}{r_s^3} \right) = \frac{T_x}{m}$ $\ddot{y} + 2\omega \dot{x} - (y + r_s) \left(\omega^2 - \frac{GM}{r_s^3} \right) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_s^3} z = \frac{T_z}{m}$ <p>EGGLESTON [4, 5]</p>
$\frac{d^2 \vec{r}_s}{dt^2} = -\omega^2 \vec{r}_s; \frac{GM}{r_s^3} = \omega^2$ On the left, $\frac{GM}{r_f^3} = \frac{GM}{r_s^3} \left(1 - 3 \frac{\vec{r}_s \cdot \vec{R}}{r_s^3} + \dots \right)$ $\frac{GM}{r_f^3} \vec{r}_f \cong \omega^2 \left(\vec{r}_s + \vec{R} - 3 \frac{\vec{r}_s \cdot \vec{R}}{r_s^3} \vec{r}_s \right)$ On the right, $\frac{GM}{r_f^3} = \frac{GM}{r_s^3} \left(1 - 3 \frac{y}{r_s} + \dots \right)$ $\frac{GM}{r_f^3} \vec{r}_f \cong \omega^2 (\vec{r}_s + \vec{R} - \vec{j} 3y)$	$\frac{d^2 \vec{R}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{R}}{dt} + \vec{\omega} \times \vec{\omega} \times \vec{R} + \frac{GM}{r_s^3} (\vec{R} - \vec{j} 3y) = \frac{\vec{T}}{m}$ <p>or</p> $\frac{d^2 \vec{R}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{R}}{dt} + \frac{GM}{r_s^3} (-\vec{j} 3y + \vec{k} z) = \frac{\vec{T}}{m}$	$\ddot{x} - 2\omega \dot{y} = \frac{T_x}{m}$ $\ddot{y} + 2\omega \dot{x} - 3\omega^2 y = \frac{T_y}{m}$ $\ddot{z} + \omega^2 z = \frac{T_z}{m}$ <p>WHEELON [14], CLOHESSEY and WILSHIRE [15], CARNEY [17], EGGLESTON [5, 18], SPRALIN [19]</p>
$\frac{GM}{r_f^3} \vec{r}_f \cong \frac{GM}{r_s^3} (\vec{r}_s + \vec{R})$ $\frac{GM}{r_s^3} = \omega^2$	$\frac{d^2 \vec{R}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{R}}{dt} + \vec{\omega} \times \vec{\omega} \times \vec{R} + \frac{GM}{r_s^3} \vec{R} = \frac{\vec{T}}{m}$ <p>or</p> $\frac{d^2 \vec{R}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{R}}{dt} + \frac{GM}{r_s^3} \vec{R} = \frac{\vec{T}}{m}$	$\ddot{x} - 2\omega \dot{y} = \frac{T_x}{m}$ $\ddot{y} + 2\omega \dot{x} = \frac{T_y}{m}$ $\ddot{z} + \omega^2 z = \frac{T_z}{m}$ <p>HORNEY [20]</p>
	$\frac{d^2 \vec{r}_f}{dt^2} + 2\vec{\Omega} \times \frac{d\vec{r}_f}{dt} + \vec{\Omega} \times \vec{r}_f + \vec{\Omega} \times \vec{\Omega} \times \vec{r}_f = \frac{\vec{T}}{m}$ <p>for any orbit</p> $\frac{d^2 \vec{R}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{R}}{dt} + \vec{\omega} \times \vec{\omega} \times \vec{r}_f = \frac{\vec{T}}{m}$ <p>for circular orbit</p>	<p>Circular orbit</p> $\ddot{x} - 2\omega \dot{y} - \omega^2 x = \frac{T_x}{m}$ $\ddot{y} + 2\omega \dot{x} - \omega^2 (y + r_s) = \frac{T_y}{m}$ $\ddot{z} = \frac{T_z}{m}$